09 Parsial Differential Equation

Partial Differential Equation

Persamaan differensial parsial secara umum untuk orde dua

Persamaan differensial parsial secara umum untuk orde dua dalam variabel bebas x dan y dapat dinyatakan sebagai berikut :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Persamaan differensial parsial dapat diklasifikasikan tergantung dari nilai $B^2 - 4AC$.

- jika $B^2 - 4AC < 0$, maka persamaan Eliptik
- jika $B^2 - 4AC = 0$, maka persamaan Parabolik
- jika $B^2 - 4AC > 0$, maka persamaan Hiperbolik

- Jika koefisien A, B, dan C dalah fungsi x, y, dan/atau u, persamaan mungkin berubah dari satu klasifikasi menjadi klasifikasi lain pada titik bervariasi.
- Dalam teknik kimia persamaan yang sering dijumpai adalah persamaan differensial eliptik dan parabolik, sehingga kedua persamaan itulah yang akan dibahas dalam kuliah ini.

PERSAMAAN DIFFERENSIAL ELIPTIK

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Persamaan differensial eliptik terbentuk jika koefisien A dan C pada persamaan umum sama dengan 1 dan B sama dengan nol, sehingga $B^2 - 4AC < 1$.

Ada 2 type persamaan differensial eliptik yang akan dibahas, yaitu

- Persamaan Laplace

$$A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} = 0$$

- Persamaan Poisson

$$A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

PERSAMAAN LAPLACE

- Persamaan Laplace sering muncul dari penyusunan persoalan perpindahan panas dalam suatu plat.
- Bentuk paling sederhana persamaan Laplace adalah

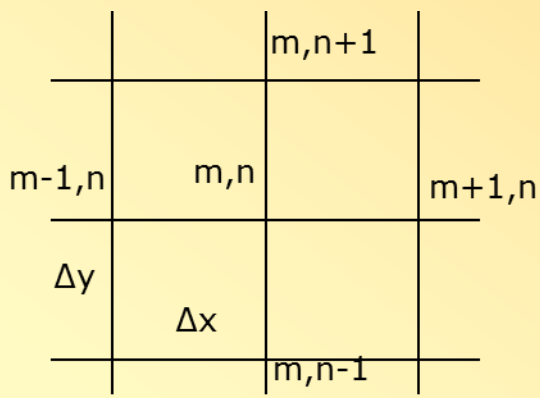
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Penyelesaian persamaan Laplace adalah metode beda hingga.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u_{i+1,j} - 2u_{i,j} + _{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + _{i,j-1}}{(\Delta y)^2}$$

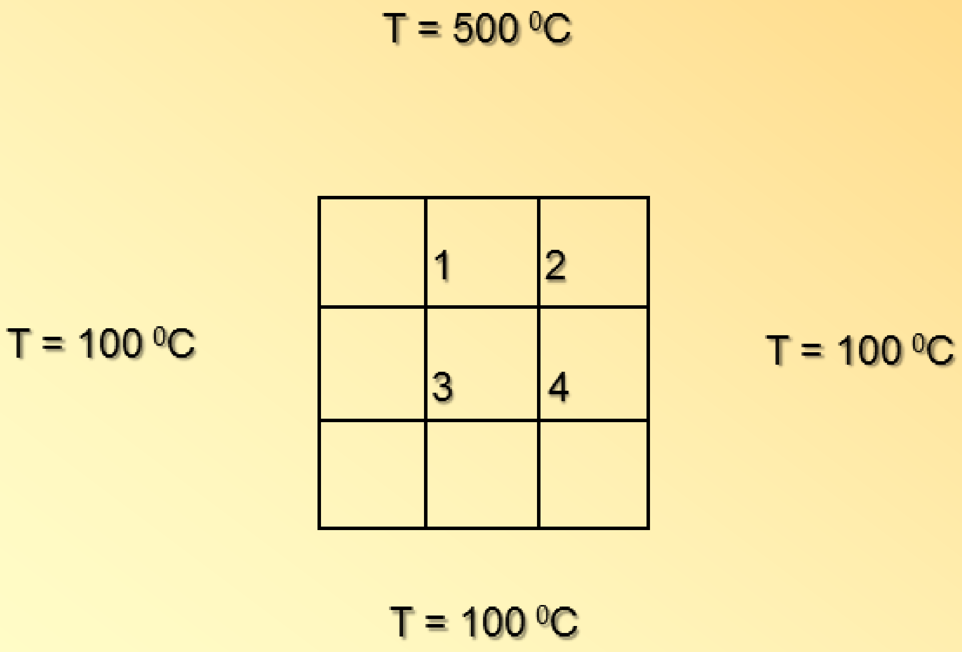
Jika diambil $\Delta x = \Delta y = h$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \right] = 0$$



Contoh 1

Plat tembaga tipis dengan ukuran 3 cm x 3 cm. Permukaan salah satu sisi dipertahankan 500 ^oC dan ketiga sisi yang lain dipertahankan pada suhu 100 ^oC. Permukaan plat diisolasi sehingga panas mengalir arah x dan y saja. Tentukan distribusi suhu plat tersebut pada keadaan tunak (*steady*).



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\left[\, u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \right] = 0$$

Contoh 2

Plat tembaga tipis dengan ukuran 6 cm x 8 cm. Permukaan salah satu sisi dengan panjang 6 cm dipertahankan 100 ^{o}C dan ketiga sisi yang lain dipertahankan pada suhu 40 ^{o}C. Permukaan plat diisolasi sehingga panas mengalir arah x dan y saja. Tentukan distribusi suhu plat tersebut pada keadaan tunak (steady).

next

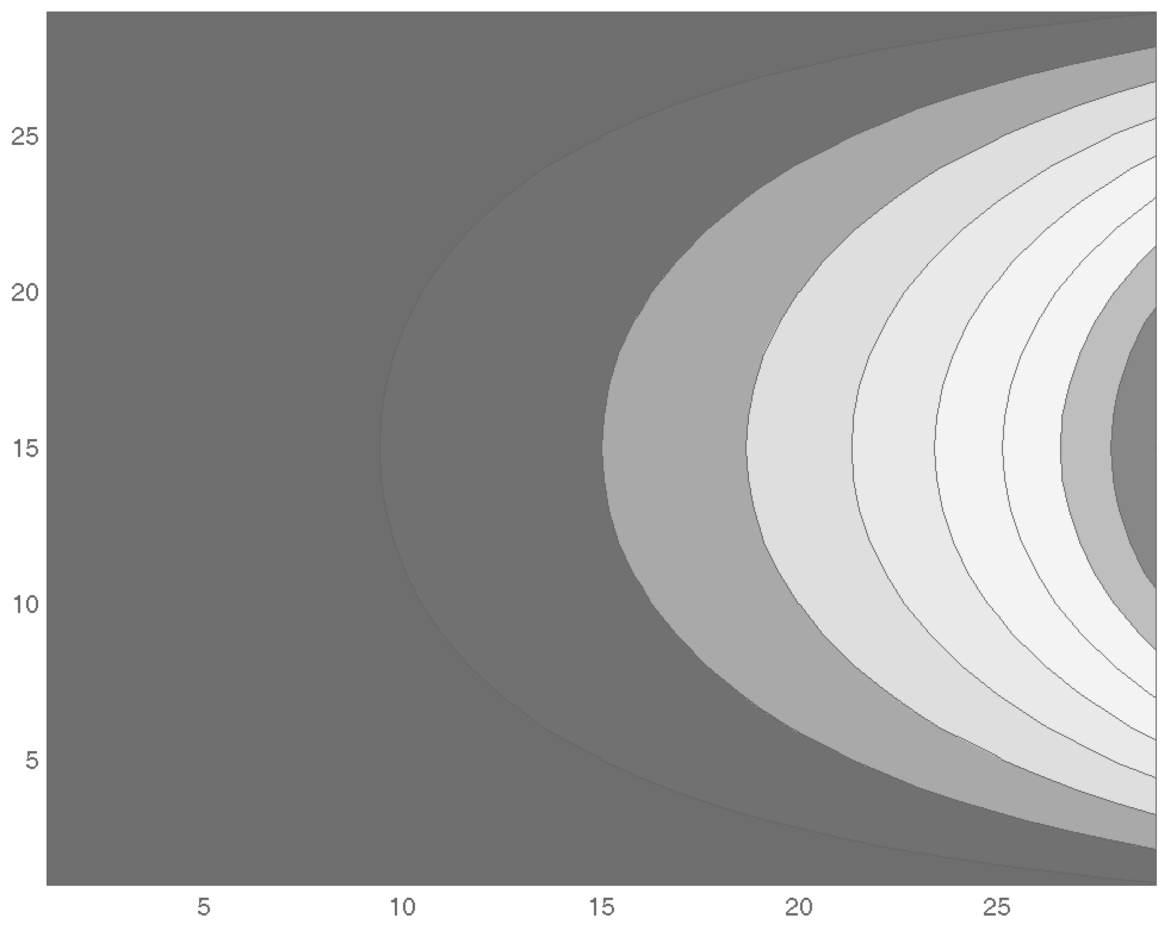
Contoh 3

Tentukan distribusi temperatur pada sebuah plat bujursangkar yang salah satu sisinya mengikuti persamaan $T = 100 \cdot \sin(\pi \cdot y)$, sedang ketiga sisi yang lain sama dengan nol.

```
clear;
N=30;

for j=0:N-2
  for i=1:N-2
     X(i+j*(N-1))=0;
  end
end
for i=1:N-1
   X(i*(N-1))=-100*sin(i*pi/N);
end
```

```
% Penyusunan matriks A
for i=1:(N-1)*(N-1)
   A(i,i)=-4;
end
for i=1:N-2
   for k=0:N-2
   A(i+k*(N-1),i+1+k*(N-1))=1;
   end
end
for k=0:N-3
   for i=1:N-1
   A(i+k*(N-1),i+(k+1)*(N-1))=1;
   end
end
for i=1:(N-1)*(N-1)
   for j=1:i
   A(i,j)=A(j,i);
   end
end
```

```
% Inversi Matriks dan Perhitungan temperatur
M=inv(A);
U=M*X';

% Plot hasil pada bentuk contour
for i=1:N-1
    for j=1:N-1
    x(i,j)=U(j+(i-1)*(N-1));
    end
end

[i,j]=meshgrid(1:1:N-1,1:1:N-1);
[c,h]=contourf(i,j,x);
```


25
20
15
10
5
5
10
15
20
25

Contoh 4

Plat tipis dari baja mempunyai ukuran 10 cm x 20 cm. Jika salah satu sisi ukuran 10 cm dijaga pada 100 ^{O}C dan ketiga sisi yang lain dijaga pada 0 ^{O}C. Tentukan profil temperatur pada plat. Untuk baja k = 0,16 kal/detik.cm^2.C/cm.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

dengan $u(x,0) = 0,$

$u(x,10) = 0,$

$u(0,y) = 0,$

$u(20,y) = 100.$

```
clear;
N=20;
M=10;

for j=0:N-2
  for i=1:M-2
     X(i+j*(M-1))=0;
  end
end
for i=1:M-1
   X(i*(N-1))=-100;
end
```

```
% Penyusunan matriks A
for i=1:(N-1)*(M-1)
   A(i,i)=-4;
end
for i=1:N-2
   for k=0:M-2
   A(i+k*(N-1),i+1+k*(N-1))=1;
   end
end
for k=0:M-3
   for i=1:N-1
   A(i+k*(N-1),i+(k+1)*(N-1))=1;
   end
end
for i=1:(N-1)*(M-1)
   for j=1:i
   A(i,j)=A(j,i);
   end
end
```

```
% Inversi Matriks dan Perhitungan Temperatur
G=inv(A);
U=G*X';

% Plot hasil bentuk contour
for i=1:M-1
    for j=1:N-1
    x(i,j)=U(j+(i-1)*(N-1));
    end
end
T = x
[i,j]=meshgrid(1:1:N-1,1:1:M-1);
[c,h]=contourf(i,j,x);
```

Fungsi ellipgen

```
function [a,om]=ellipgen(nx,hx,ny,hy,G,F,bx0,bxn,by0,byn)
%       Penyelesaian persamaan PD Parsial Eliptik
%         d^2 Z   d^2 Z
%         ----- + ----- + G(x,y)*Z = F(x,y)
%         d x^2   d y^2
%       pada plat rektanguler
%       Cara menggunakan fungsi ini
%       hx,hy = ukuran titik arah x, y
%       F, G = array (ny+1,nx+1) representasi F(x,y), G(x,y)
%       bx0, bxn = vektor baris kondisi batas pada x0, xn
%       by0, byn = vektor baris kondisi batas pada y0, yn
%       a = array (ny+1,nx+1) penyelesaian
%
%       Nama File : ellipgen.m
%       Surakarta, Oktober 2005
%   ---------------------------------------------------------------

nmax=(nx-1)*(ny-1); r=hy/hx;
a=zeros(ny+1,nx+1); p=zeros(ny+1,nx+1);
if nargin==6
```

```
nmax=(nx-1)*(ny-1); r=hy/hx;
a=zeros(ny+1,nx+1); p=zeros(ny+1,nx+1);
if nargin==6
    ncase=0;mode=F;
end
if nargin==10
    test=0;
    if F==zeros(ny+1,nx+1), test=1; end
    if bx0==zeros(1,ny+1), test=test+1; end
    if bxn==zeros(1,ny+1), test=test+1; end
    if by0==zeros(1,nx+1), test=test+1; end
    if byn==zeros(1,nx+1), test=test+1; end
    if test==5
        disp(' WARNING ')
        disp('  ')
        break
    end
    bx0=bx0(1,ny+1:-1:1); bxn=bxn(1,ny+1:-1:1);
    a(1,:)=byn; a(ny+1,:)=by0;
    a(:,1)=bx0'; a(:,nx+1)=bxn';ncase=1;
end
for i=2:ny
    for j=2:nx
        nn=(i-2)*(nx-1)+(j-1);
        q(nn,1)=i; q(nn,2)=j; p(i,j)=nn;
    end
end
C=zeros(nmax,nmax); e=zeros(nmax,1); em=zeros(nmax,1);
if ncase==1, g=zeros(nmax,1); end
for i=2:ny
    for j=2:nx
        nn=p(i,j); C(nn,nn)=-(2+2*r^2); e(nn)=hy^2*G(i,j);
        if ncase==1, g(nn)=g(nn)+hy^2*F(i,j); end
        if p(i+1,j)~=0
            np=p(i+1,j); C(nn,np)=1;
        else
            if ncase==1, g(nn)=g(nn)-by0(j); end
        end
        if p(i-1,j)~=0
            np=p(i-1,j); C(nn,np)=1;
```

Distribusi Temperatur pada Plat Rektanguler

Tentukan distribusi temperatur dalam suatu plat rektanguler, dengan kondisi batas sebagai berikut

$x = 0, T = 100y$

$x = 3, T = 250y$

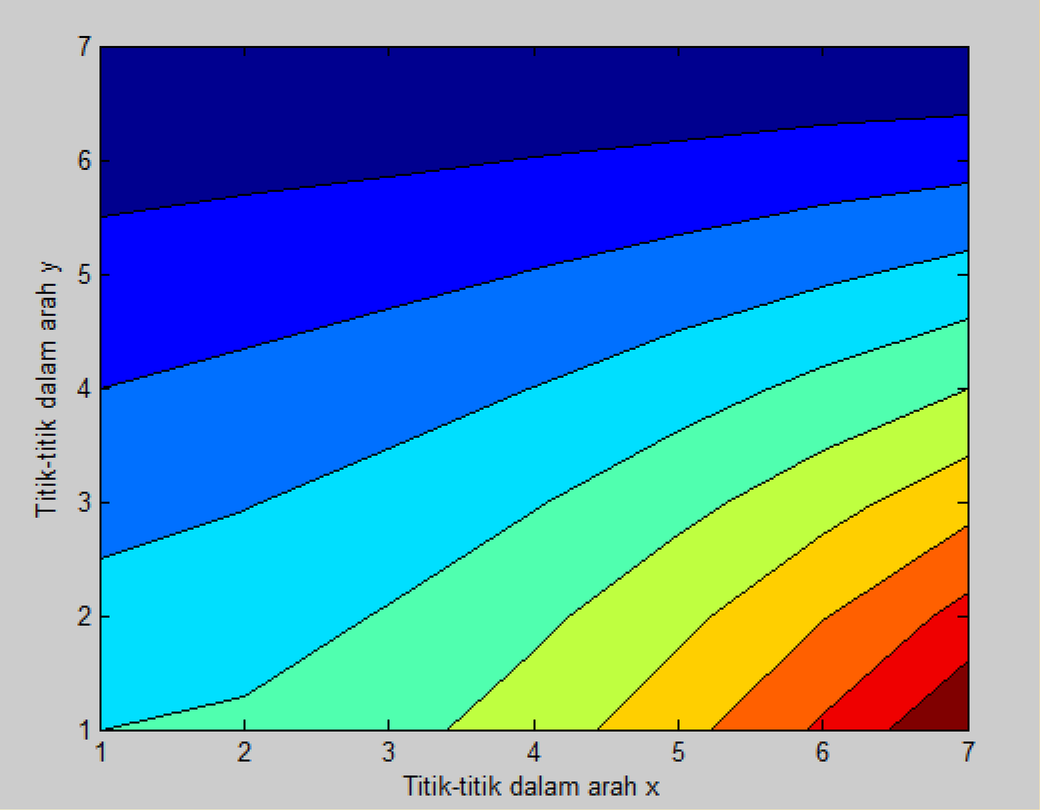
$y = 0, T = 0$

$y = 2, T = 200 + (100/3)x^2$

Penyelesaian untuk ukuran 6 x 6.

Program Matlab

```
clear all
% Jumlah titik arah x, y
nx=6; ny=6;
% Ukuran titik arah x, y
hx=0.5; hy=0.3333;
% Input data pada kondisi batas
by0=[0 0 0 0 0 0 0];
byn=[200 208.33 233.33 275 333.33 408.33 500];
bx0=[0 33.33  66.67 100 133.33 166.67 200];
bxn=[0 83.33 166.67 250 333.33 416.67 500];
% Penyelesaian dg fungsi ellipgen
F=zeros(ny+1,nx+1); G=F;                % PD Laplace
a=ellipgen(nx, hx, ny, hy, G, F, bx0, bxn, by0, byn);
% Plot hasil
contourf(a)
xlabel('Titik-titik dalam arah x');
ylabel('Titik-titik dalam arah y');
```

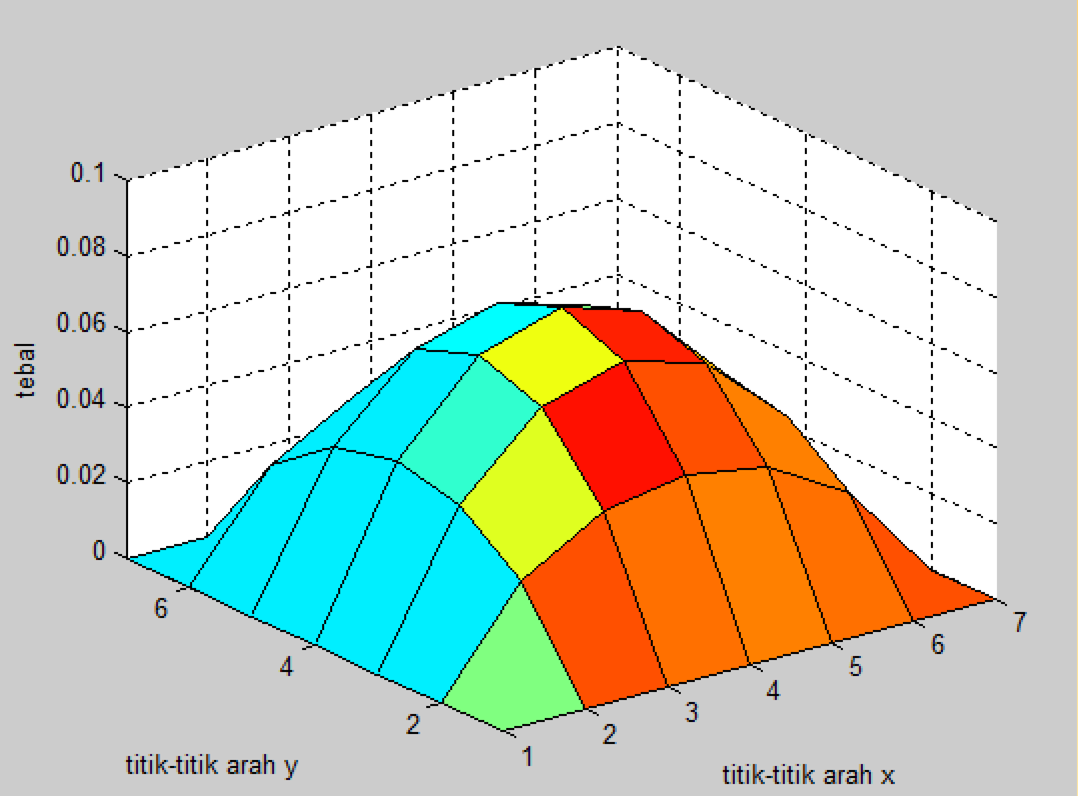

7
6
5
4
3
2
1
1
2
3
4
5
6
7
Titik-titik dalam arah x
Titik-titik dalam arah y

PERSAMAAN POISSON

$$A\frac{\partial^2 u}{\partial x^2}+C\frac{\partial^2 u}{\partial y^2}=F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

Tentukan defleksi membran bujursangkar seragam dengan ujung-ujung tetap dijaga. Sedang beban distribusi dapat didekati dengan suatu beban pada suatu titik. Permasalahan ini mengikuti persamaan Poisson dengan F(x,y) menunjukkan beban membran.

```
% Jumlah titik arah x, y
nx=6; ny=6;
% Ukuran titik arah x, y
hx=1/6; hy=1/6;
% Input data pada kondisi batas
by0=[0 0 0 0 0 0 0];
byn=[0 0 0 0 0 0 0];
bx0=[0 0 0 0 0 0 0];
bxn=[0 0 0 0 0 0 0];
% Penyelesaian dg fungsi ellipgen
F=-ones(ny+1,nx+1); G=zeros(nx+1,ny+1);
a=ellipgen(nx, hx, ny, hy, G, F, bx0, bxn, by0, byn);
% Plot hasil
surfl(a)
axis([1 7 1 7 0 0.1])
xlabel('titik-titik arah x');
ylabel('titik-titik arah y ');
zlabel('tebal');
```



tebal
0.1
0.08
0.06
0.04
0.02
0
6
4
2
titik-titik arah y
1
2
3
4
5
6
7
titik-titik arah x

PERSAMAAN PARABOLIK

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{k}\frac{\partial u}{\partial t}$$

METODE EKSPLISIT

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^{j} - 2u_i^j + u_{i-1}^{j}}{(\Delta x)^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^j}{\Delta t}$$

$$u_i^{j+1} = \frac{k\Delta t}{c\rho(\Delta x)^2}\left(u_{i+1}^j + u_{i-1}^j\right) + \left(1 - \frac{2k\Delta t}{c\rho(\Delta x)^2}\right)u_i^j$$

Penyederhanaan $\frac{k\Delta t}{c\rho(\Delta x)^2} = \frac{1}{2} = \frac{1}{M}$

$$u_i^{j+1} = \frac{1}{2}\left(u_{i+1}^j + u_{i-1}^j\right)$$

Distribusi Temperatur sebagai Fungsi Waktu pada Plat Tipis

Plat besi yang sangat luas mempunyai tebal 2 cm. Temperatur mula-mula dalam plat merupakan fungsi jarak dari salah satu sisinya sebagai berikut :

$u = 100x$ untuk $0 < x < 1$,

$u = 100(2 - x)$ untuk $1 < x < 2$.

Tentukan temperatur tebal plat sebagai fungsi x dan t, jika kedua permukaan tetap dijaga 0 ^{O}C. Untuk besi k = 0,13 kal/detik.cm. ^{O}C, c = 0,11 cal/g. ^{O}C, ρ = 7,8 g/cm^3.

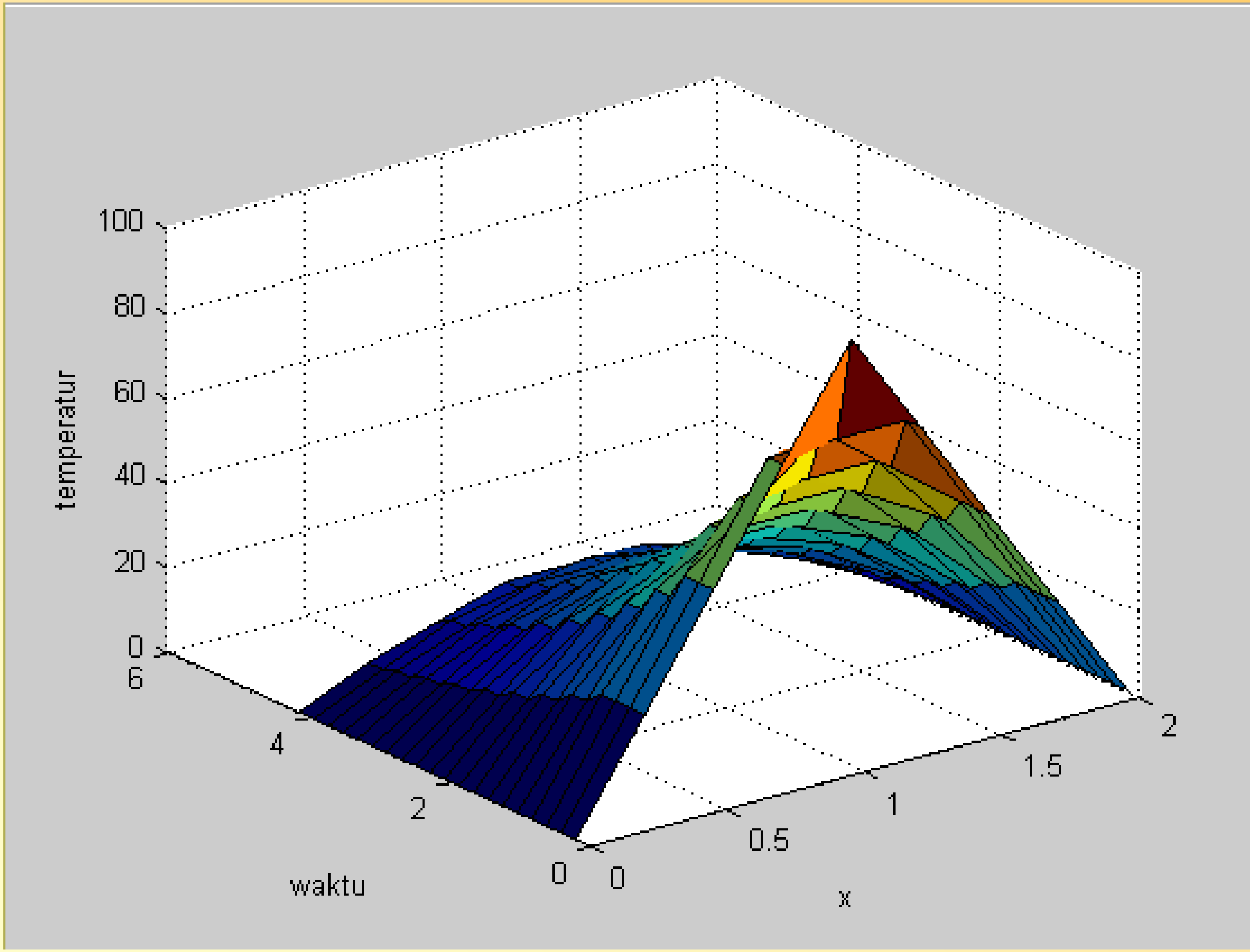
$$\frac{k\Delta t}{c\rho(\Delta x)^2} = \frac{1}{2} = \frac{1}{M}$$

$\Delta x = 0{,}25$ sehingga $\Delta t = 0{,}206$ detik

```
clc
clear all

% Data-data
L=2;
k=0.13;
c=0.11;
rho=7.8;

% Interval
N=8;
M=0.5;
delx=L/N;
delt=M*c*rho*delx^2/k;
xo=0;
Jend=20;
```

```
for i=1:N+1
    x(i)=xo+delx*(i-1);
end

% Kondisi awal
for i=1:ceil((N+1)/2)
    u(1,i)=100*x(i);
end
for i=N+1:-1:ceil((N+1)/2)
    u(1,i)=100*(2-x(i));
end
for i=1:Jend
    t(i)=delt*i;
end
```

```
for j=1:Jend
    for i=2:ceil((N+1)/2)+1
        u(j+1,i)=(k*delt/(c*rho*delx^2))*(u(j,i-1)+u(j,i+1))+...
            (1-2*k*delt/(c*rho*delx^2))*u(j,i);
    end
    for i=N+1:-1:ceil((N+1)/2)+1
        u(j+1,i)=u(j+1,ceil((N+1)/2)*2-i);
    end
end
t', u
surfl(x, t', u(1:Jend+1,:))
xlabel('x');  ylabel('waktu');
zlabel(‘konsentrasi')
```



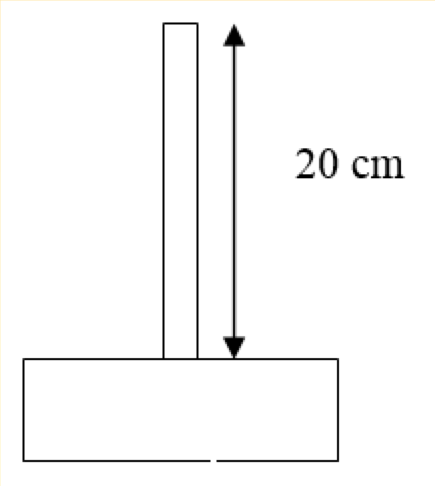
100
80
60
40
20
0
temperatur
6
4
2
0
waktu
0
0.5
1
1.5
2
x

next

Difusi alkohol

Suatu tabung panjang 20 cm mula-mula berisi udara dengan 2 % uap alkohol. Pada bagian bawah tabung berhubungan dengan bejana berisi alkohol sehingga alkohol tersebut menguap melalui tabung yang mula-mula berisi udara diam tersebut. Pada bagian ini konsentrasi alkohol dijaga tetap 10 %. Pada bagian atas (puncak) tabung uap alkohol di permukaan atas tabung dapat dianggap selalu nol.

Tentukan distribusi konsentrasi alkohol pada tabung sampai minimal 1000 detik. Diketahui ϑ = 0,119 cm^2 / detik.



Persamaan Parabolik

```
     d^2 c      d c
D -------- =  -----
     d x^2      d t
```

Kondisi awal

c(x,0) = 2

Kondisi batas

c(0,t) = 0   c(20,t) = 10

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{(\Delta x)^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^{j}}{\Delta t}$$

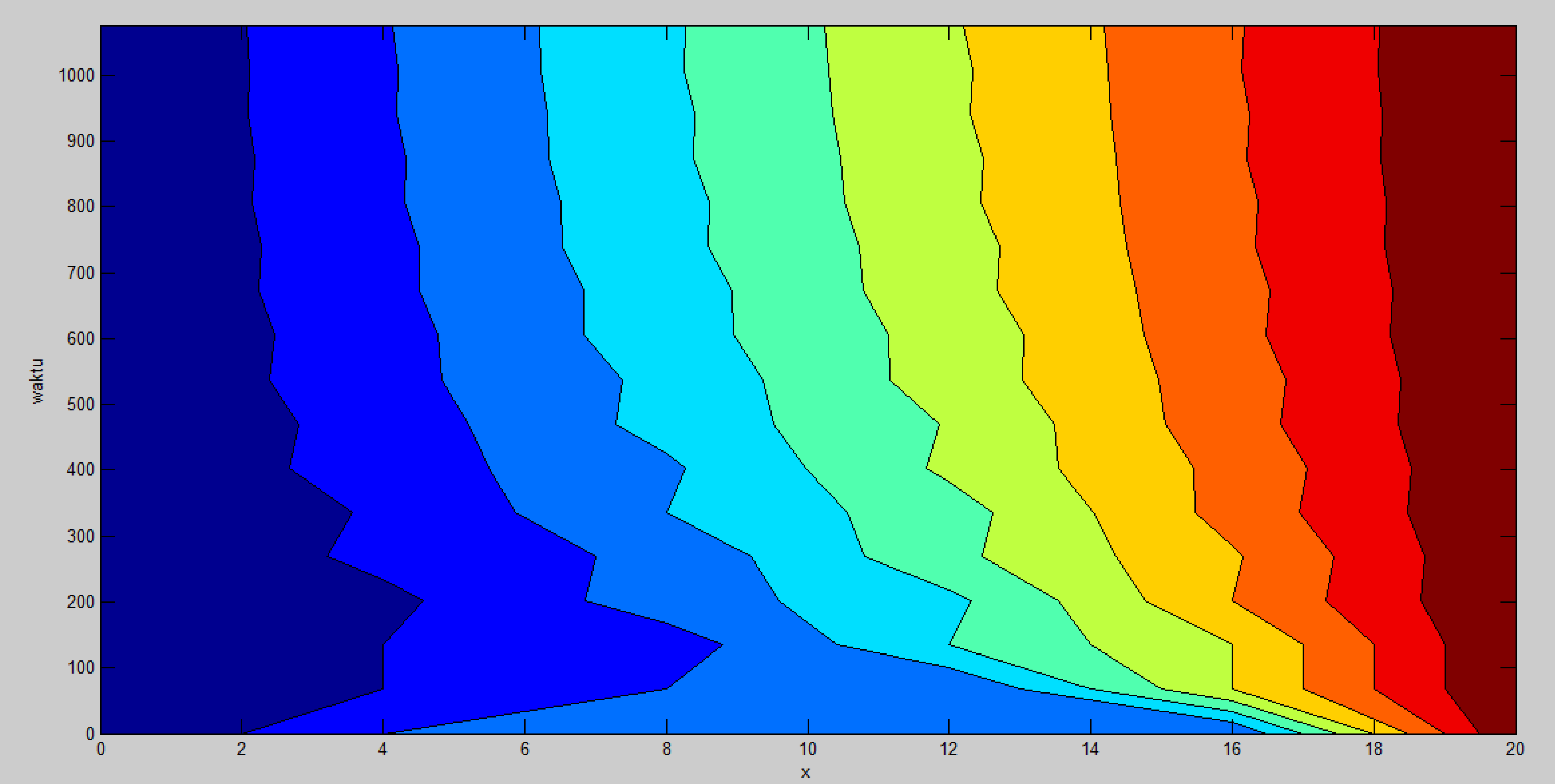
$r = \vartheta\ \Delta t/(\Delta x)^2 = 1/2$ dan $\Delta x = 4$ cm.

$\Delta t = 0.5(\Delta x)^2/\vartheta = 67{,}2$ detik

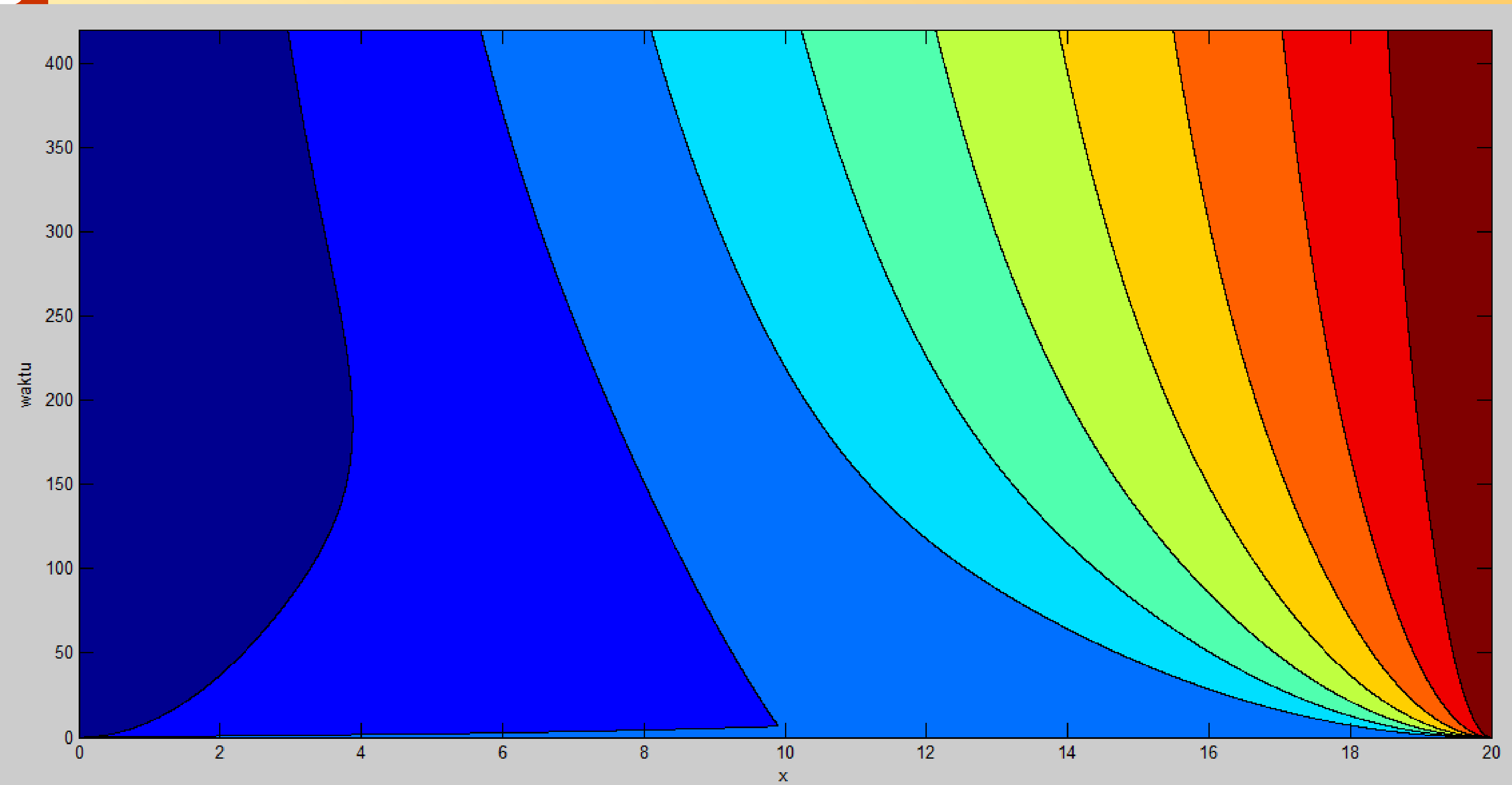
```
format short
clc
clear all
% Data-data
L=20;
D=0.119;
N=5;
M=0.5;
delx=L/N;
delt=M/D*delx^2;
xo=0;
Jend=16;
for i=1:N+1
        x(i)=xo+delx*(i-1);
end
x
%Kondisi awal
u(1,1)=0.0;
for i=2:N
   u(1,i)=2;
end
u(1,N+1)=10.0;
% interval waktu
for i=1:Jend+1
   t(i) = delt*i-delt;
end
t=t'
for j=1:Jend
   u(j+1,1)=0.0;
   for i=2:N

u(j+1,i)=(delt*D/delx^2)*(u
(j,i-1)+u(j,i+1))+...
(1-2*delt*D/delx^2)*u(j,i);
   end
   u(j+1,N+1)=10.0;
end
u
mesh(x, t, u(1:Jend+1,:))
xlabel('x');  ylabel('waktu');
zlabel('konsentrasi')
```

```
x =
     0     4     8    12    16    20

t =
  1.0e+003 *
         0
    0.0672
    0.1345
    0.2017
    0.2689
    0.3361
    0.4034
    0.4706
    0.5378
    0.6050
    0.6723
    0.7395
    0.8067
    0.8739
    0.9412
    1.0084
    1.0756

u =
         0    2.0000    2.0000    2.0000    2.0000   10.0000
         0    1.0000    2.0000    2.0000    6.0000   10.0000
         0    1.0000    1.5000    4.0000    6.0000   10.0000
         0    0.7500    2.5000    3.7500    7.0000   10.0000
         0    1.2500    2.2500    4.7500    6.8750   10.0000
         0    1.1250    3.0000    4.5625    7.3750   10.0000
         0    1.5000    2.8438    5.1875    7.2813   10.0000
         0    1.4219    3.3438    5.0625    7.5938   10.0000
         0    1.6719    3.2422    5.4688    7.5313   10.0000
         0    1.6211    3.5703    5.3867    7.7344   10.0000
         0    1.7852    3.5039    5.6523    7.6934   10.0000
         0    1.7520    3.7188    5.5986    7.8262   10.0000
         0    1.8594    3.6753    5.7725    7.7993   10.0000
         0    1.8376    3.8159    5.7373    7.8862   10.0000
         0    1.9080    3.7875    5.8511    7.8687   10.0000
         0    1.8937    3.8795    5.8281    7.9255   10.0000
         0    1.9398    3.8609    5.9025    7.9140   10.0000
```

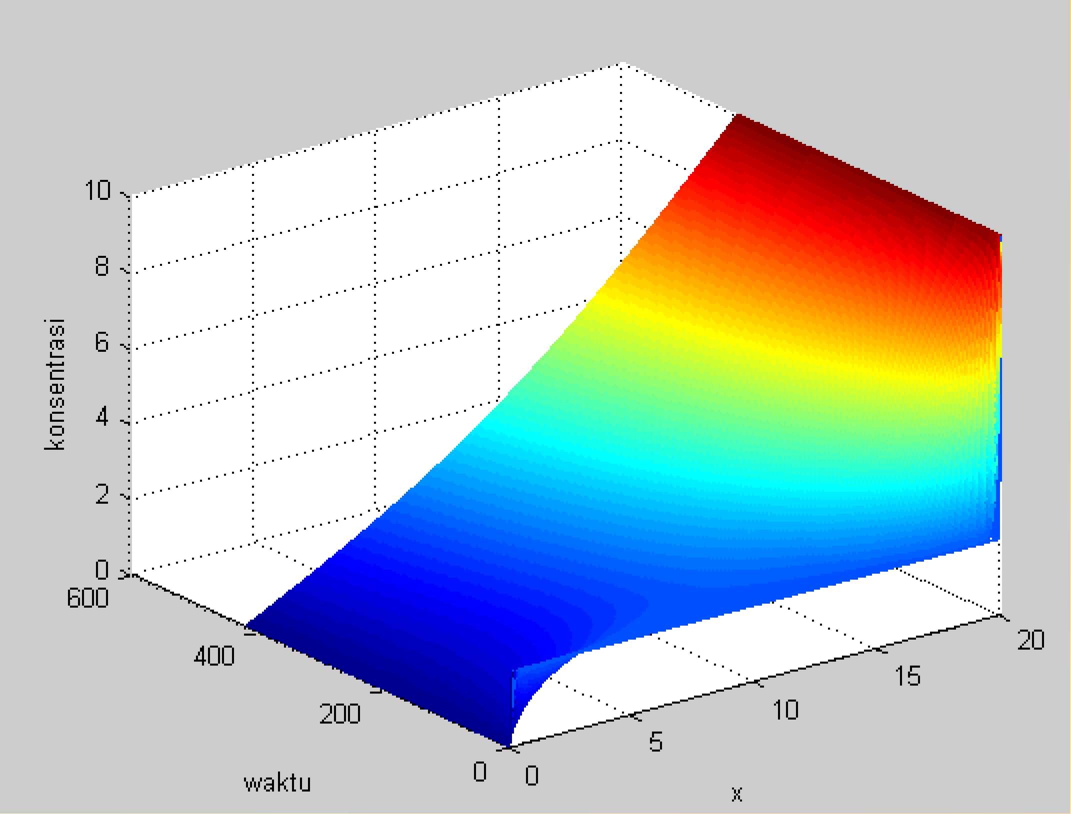
09 Parsial Differential Equation



- Coba untuk $\Delta x = 0.1$ cm



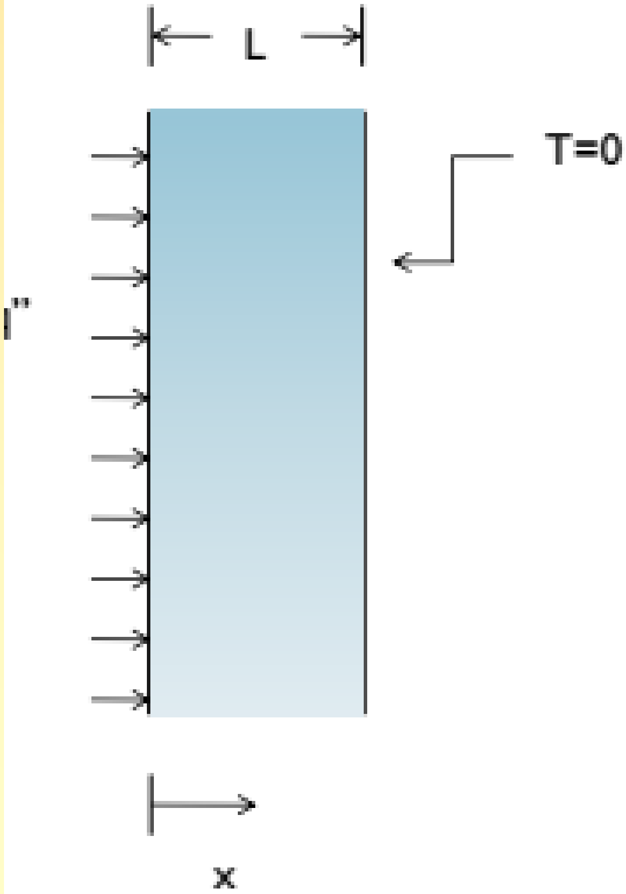
waktu
400
350
300
250
200
150
100
50
0
0
2
4
6
8
10
12
14
16
18
20
x


10
8
6
4
2
0
konsentrasi
600
400
200
0
waktu
0
5
10
15
20
x

next

PDE tool

Model Problem



$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = 0$$

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = q''$$

$$T(L,t) = 0$$

pdepe Solves the Following

m=0 for Cartesian, 1 for cylindrical, 2 for spherical

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}=x^{-m}\frac{\partial}{\partial x}\left(x^{m}f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right)+s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$u(x,t_0)=u_0(x)$$

Initial Conditions

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

Boundary Conditions – one at each boundary

pdepe Solves the Following

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$c = \rho c_p$$

$$f = k\frac{\partial T}{\partial x}$$

$$s = 0$$

Differential Equations

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
```

Initial Conditions

```
function u0 = pdex1ic(x)
u0 = 0;
```

pdepe Solves the Following

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q'' \qquad p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

or

$$q''+k\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

$x=0$

$p=q''$

$q=1$

remember

$$f=k\frac{\partial T}{\partial x}$$

$x=L$

$p=T=ur$

$q=0$

$$T(L,t)=0$$

Boundary Conditions

```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q;
ql = 1;
pr = ur;
qr = 0;
```

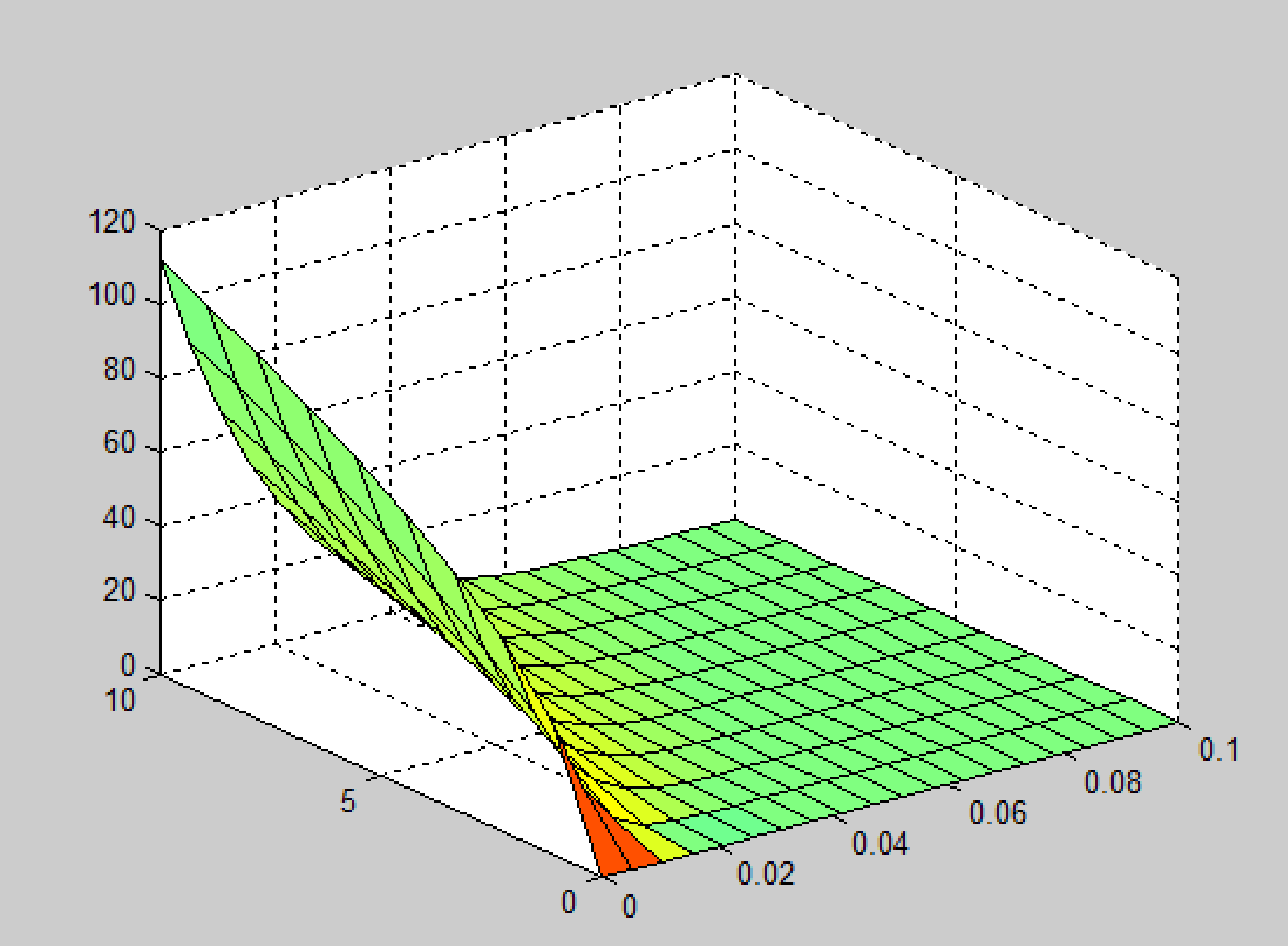
At left edge, q+k*dT/dx=0

At right edge, ur=0

```
function parabolic
global rho cp k
global q
L=0.1 %m
k=200 %W/m-K
rho=10000 %kg/m^3
cp=500 %J/kg-K
q=1e6 %W/m^2
tend=10 %seconds
m = 0;
x = linspace(0,L,20);
t = linspace(0,tend,10);
sol = pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
surfl(x,t,sol)
```

```
-------------------------------------------------------------
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
-------------------------------------------------------------
function u0 = pdex1ic(x)
u0 = 0;
-------------------------------------------------------------
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q; %these two set k*dT/dx-q=0 on right side
ql = 1;
pr = ur;
qr = 0; %sets right side temperature to 0
```

09 Parsial Differential Equation



PDEtool

equation solver such as **ode45**. The following specific PDE can be solved with **pdepe**:

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left[x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right] + s\left(x,t,u,\frac{\partial u}{\partial x}\right) \quad (11.3.1)$$

In addition to the PDE, boundary conditions must also be specific. The specific form required for **pdepe** can be nonlinear and time dependent so that:

$$p(x,t,u) + q(x,t)g(x,t,u,u_x) = 0 \text{ at } x = a, b\,. \quad (11.3.2)$$

PDEtool

To start, consider the simplest PDE: the heat equation:

$$\pi^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{11.3.3}$$

where the solution is defined on the domain $x \in [0, 1]$ with the boundary conditions

$$u(0, t) = 0 \tag{11.3.4a}$$

$$\pi \exp(-t) + \frac{\partial u(1, t)}{\partial t} = 0 \tag{11.3.4b}$$

and initial conditions

$$u(x, 0) = \sin(\pi x)\,. \tag{11.3.5}$$

M-file : Main command

```
m = 0;
x = linspace(0,1,20);
t = linspace(0,2,5);
u = pdepe(m,'pdex1pde','pdex1ic','pdex1bc',x,t);
surf(x,t,u)
```

Function pdex1pde

$$\pi^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

equation solver such as **ode45**. The following specific PDE can be solved with **pdepe**:

$$c\left(x, t, u, \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial t} = x^{-m} \frac{\partial}{\partial x}\left[x^m f\left(x, t, u, \frac{\partial u}{\partial x}\right)\right] + s\left(x, t, u, \frac{\partial u}{\partial x}\right) \quad (11.3.1)$$

In this case, the implementation is fairly straightfoward since $s(x, t, u, u_x) = 0$, $m = 0$, $c(x, t, u, u_x) = \pi^2$ and $f(x, t, u, u_x) = u_x$. The initial condition is quite easy and can be done in one line.

pdex1pde.m

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
c = pi^2;
f = DuDx;
s = 0;
```

The initial condition is quite easy and can be done in one line

and initial conditions

$$u(x, 0) = \sin(\pi x)\,.$$

pdex1ic.m

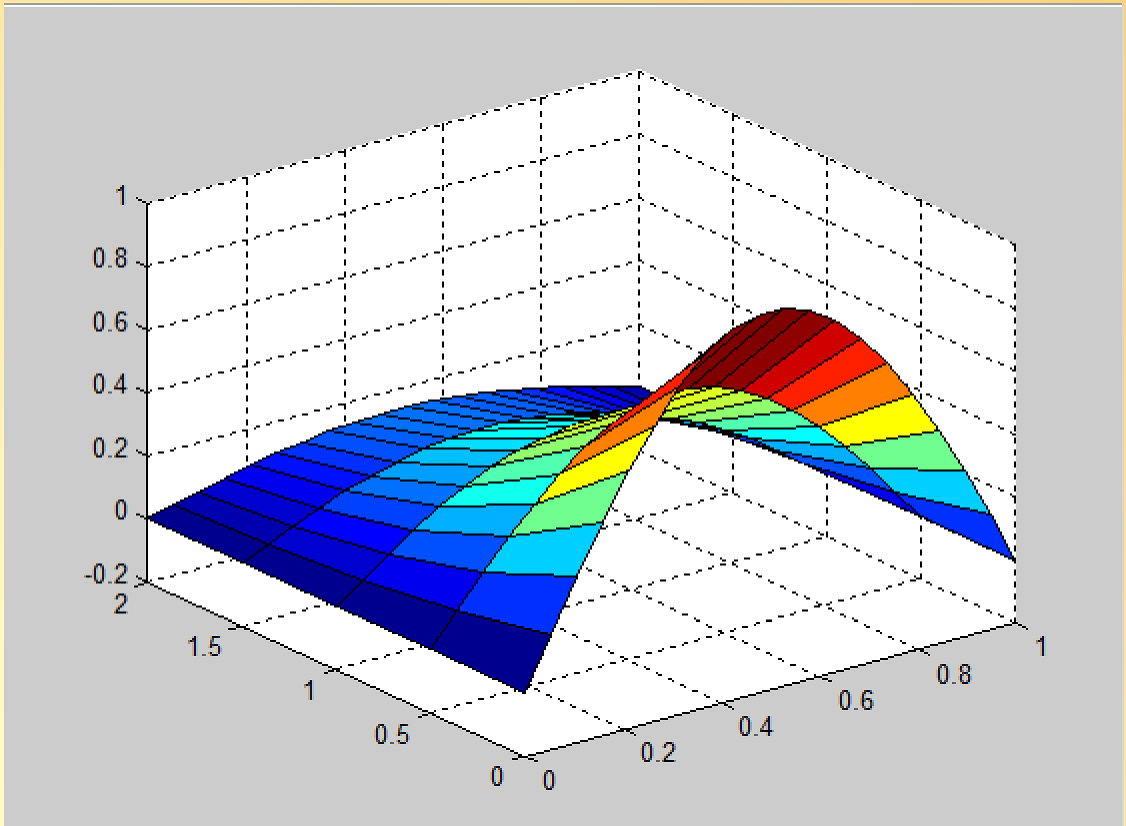
```
function u0 = pdex1ic(x)
u0 = sin(pi*x);
```

the left and right boundaries can be implemented by specifying the function q, q and g at the right and left

$$u(0,t) = 0$$

$$\pi \exp(-t) + \frac{\partial u(1,t)}{\partial t} = 0$$

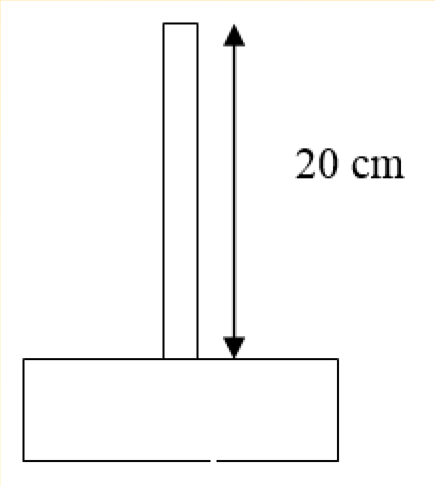
```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr = pi * exp(-t);
qr = 1;
```


1
0.8
0.6
0.4
0.2
0
-0.2
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

- Selesaikan soal sebelumnya dg PDEtools

Difusi alkohol

Suatu tabung panjang 20 cm mula-mula berisi udara dengan 2 % uap alkohol. Pada bagian bawah tabung berhubungan dengan bejana berisi alkohol sehingga alkohol tersebut menguap melalui tabung yang mula-mula berisi udara diam tersebut. Pada bagian ini konsentrasi alkohol dijaga tetap 10 %. Pada bagian atas (puncak) tabung uap alkohol di permukaan atas tabung dapat dianggap selalu nol.



Tentukan distribusi konsentrasi alkohol pada tabung sampai minimal 1000 detik. Diketahui $\vartheta = 0{,}119\ \text{cm}^2$ / detik.

Persamaan Parabolik

$$D \frac{d^2 c}{d x^2} = \frac{d c}{d t}$$

Kondisi awal

$c(x,0) = 2$

Kondisi batas

$c(0,t) = 0$ $\quad$ $c(20,t) = 10$

pdepe Solves the Following

m=0 for Cartesian, 1 for cylindrical, 2 for spherical

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}=x^{-m}\frac{\partial}{\partial x}\left(x^{m}f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right)+s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$u(x,t_0)=u_0(x)$$

Initial Conditions

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

Boundary Conditions – one at each boundary

The Main file

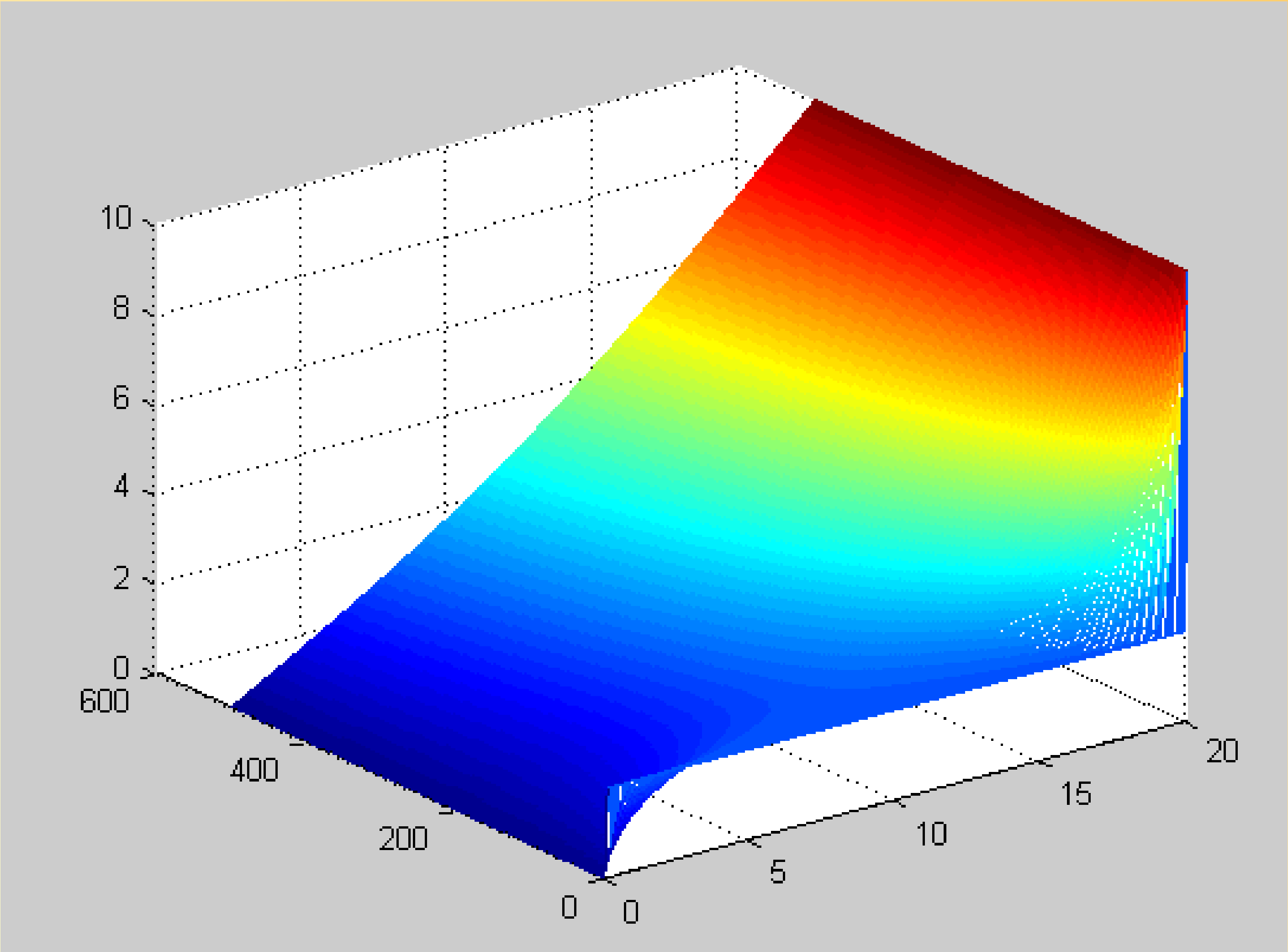
```
m = 0;
x = linspace(0, 20, 201);
t = linspace(0, 500, 501)
u = pdepe(m,'pdex3pde','pdex3ic','pdex3bc',x,t);
mesh(x,t,u)
```

```
function [c,f,s] = pdex3pde(x,t,u,DuDx)
c = 1/0.119;
f = DuDx;
s = 0;
```

```
function u0 = pdex3ic(x)
u0 = 2;
```

```
function [pl,ql,pr,qr] = pdex3bc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr = ur-10;
qr = 0;
```

09 Parsial Differential Equation



the transient behavior of a rod at constant T put between two heat reservoirs at different temperatures, again T1 = 100, and T2 = 200. The rod will start at 150. Over time, we should expect a solution that approaches the steady state solution: a linear temperature profile from one side of the rod to the other.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

t = 0; To(x) = 150

T(0, t) = 100

T(L,t) = 200

```
m=0; % specifies 1-D symmetry
x = linspace(0,1); % equally-spaced points along the length of the rod
t = linspace(0,5); % this creates time points
sol = pdepe(m,@pdex,@pdexic,@pdexbc,x,t);
surf(x,t,sol)
xlabel('Position')
ylabel('time')
zlabel('Temperature')
```

```
function [c,f,s] = pdex(x,t,u,DuDx)
c = 1;
f = 0.02*DuDx;
s = 0;



function u0 = pdexic(x)
u0 = 150;



function [pl,ql,pr,qr] = pdexbc(xl,ul,xr,ur,t)
pl = u1-100;
ql = 0;
pr= ur-200;
qr = 0;
```

A Second Problem

- Suppose we want convection at x=L
- That is

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

$$-k\frac{dT}{dx}=h(T-T_{bulk})$$

or

$$hT-hT_{bulk}+k\frac{dT}{dx}=0$$

$$x=L$$

$$p=T=h(ur-T_{bulk})$$

$$q=1$$

Diffusion Equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

Initial Condition

$$C(x,0) = 0$$

Boundary Conditions

$$C(0,t) = C_{max}$$

$$\frac{\partial c}{\partial x}(L,t) = 0$$

The diffusion equation as derived earlier can be written as

$$\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} \qquad (37.4)$$

Initial condition

$$C(x, t = 0) = 0 \qquad (37.5)$$

Boundary conditions

$$C(x = 0, t) = C_o \qquad (37.6)$$

$$C(x = \infty, t) = 0 \qquad (37.7)$$

Comparing (37.4) with (37.1), we have

$$g\left(x, t, c, \frac{\partial c}{\partial x}\right) = 1 \qquad (37.6)$$

$$f\left(x, t, c, \frac{\partial c}{\partial x}\right) = D_x \frac{\partial c}{\partial x} \qquad (37.7)$$

$$s\left(x, t, c, \frac{\partial c}{\partial x}\right) = 0 \qquad (37.8)$$

$$m = 1 \qquad (37.9)$$

Comparing (37.5) with (37.2), we have

$$\mathrm{co} = 0 \qquad (37.10)$$

Comparing (37.6) and (37.7) with (37.2), we have

$$pl = cl - 1 \qquad (37.11)$$

$$ql = 0 \qquad (37.12)$$

$$pr = cr \qquad (37.13)$$

$$qr = 0 \qquad (37.14)$$

```
function [g,f,s] = pdefun_ex1(x,t,c,DcDx)
      D=0.0000972;
      g = 1;
      f = D*DcDx;
      s = 0;
end
```

Function icfun

```
function c0 = icfun_ex1(x)
      c0 = 0;
end
```

Function bcfun

```
function [pl,ql,pr,qr] = bcfun_ex1(xl,cl,xr,cr,t)
      pl = cl-1;
      ql = 0;
      pr = cr;
      qr = 0;
end
```

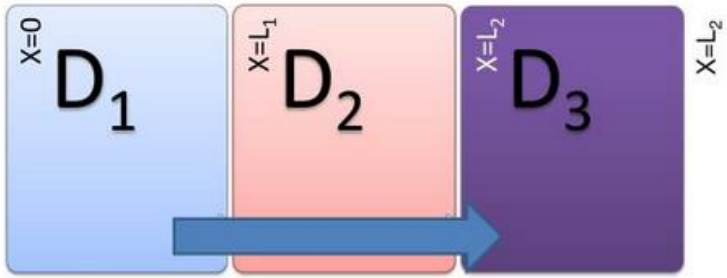
```
clear all;
Close all;

m = 0;
x = linspace(0,2.5,200); % Spatial discretization
t = linspace(0,730,100); % Temporal descretization

sol = pdepe(m,@pdefun_ex1,@icfun_ex1,@bcfun_ex1,x,t);

% The following loop is for plotting the result

for i=10:10:100
    plot(x,sol(i,:),'color', [rand rand rand]);
    axis tight;
    xlabel('X');
    ylabel('C/Co');
    hold on;
end
```

```
%---------------------functions------------------------------
function [c,f,s] = transfun(x,t,u,DuDx,D,v,c0,cin)
c = 1;
f = D*DuDx;
s = -v*DuDx;
% -----------------------------------------------------------
function u0 = ictransfun(x,D,v,c0,cin)
u0 = c0;
% -----------------------------------------------------------
function [pl,ql,pr,qr] = bctransfun(xl,ul,xr,ur,t,D,v,c0,cin)
pl = ul-cin;
ql = 0;
pr = 0;
qr = 1;
```

09 Parsial Differential Equation

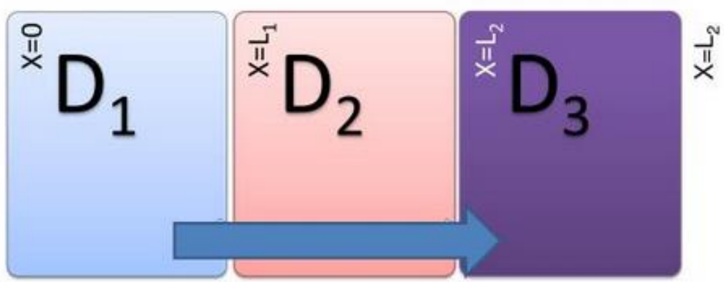
```
function pdepetrans
% transport-solver using 'pdepe'

T = 4.;                         % maximum time [s]
L = 2.5;                        % length [m]
D = 0.01;                       % diffusivity [m*m/s]
v = 1;                          % velocity [m/s]
c0 = 0;                         % initial concentration [kg/m*m*m]
cin = 1;                        % boundary concentration [kg/m*m*m]
M = 100;                        % number of timesteps
N = 100;                        % number of nodes

t = linspace (T/M,T,M);         % time discretization
x = linspace (0,L,N);           % space discretization

%----------------------execution---------------
options = odeset;
c = pdepe (0,@transfun,@ictransfun,@bctransfun,x,…
 [0 t],options,D,v,c0,cin);

%--------------------- output ---------------
plot ([0 t],c)                   % breakthrough curves
      xlabel ('time'); ylabel ('concentration');
```

Example 1: A Mass Transfer System



$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

Initial Conditions

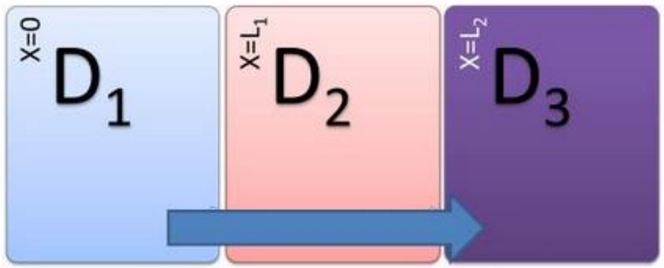


@ t=0

$0 \le x \le L_1$	$L_1 \le x \le L_2$	$L_2 \le x \le L_3$
$C = C_0$	$C = 0$	$C = 0$

Boundary Conditions

$$x = 0 \qquad \frac{\partial C}{\partial x} = 0$$



$$x = L_3 \qquad \frac{\partial C}{\partial x} = 0$$

$$x = L_1 \qquad -D_1 \frac{\partial C}{\partial x} = -D_2 \frac{\partial C}{\partial x}$$

$$x = L_2 \qquad -D_2 \frac{\partial C}{\partial x} = -D_3 \frac{\partial C}{\partial x}$$

7

$$c(x,t,u,\frac{\partial u}{\partial x})\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}(x^m b(x,t,u,\frac{\partial u}{\partial x})) + s(x,t,u,\frac{\partial u}{\partial x})$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

$$c(x,t,u,u_x) = 1$$

$$m = 0$$

$$b(x,t,u,u_x) = D_1.\frac{\partial C}{\partial x} = D_1.u_x$$

$$s(x,t,u,u_x) = 0$$

$$x = 0, \quad \frac{\partial C}{\partial x} = 0$$

$$p(xl, t, u) + q(xl, t).b(xl, t, u, u_x) = 0$$

$$b = D_1 \cdot \frac{\partial C}{\partial x}$$

$$\boxed{p(0, t, u) = 0} \qquad \boxed{q(0, t) = \frac{1}{D_1}}$$

$$x = L_1, \quad -D_1 \frac{\partial C}{\partial x} = -D_2 \frac{\partial C}{\partial x} \text{ or}$$

$$x = L_1, \quad \frac{(D_2 - D_1)}{D_1} D_1 \frac{\partial C}{\partial x} = 0$$

$$\boxed{D_1 \cdot \frac{\partial C}{\partial x}}$$

$$p(xr, t, u) + q(xr, t).b(xr, t, u, u_x) = 0$$

$$\boxed{p(L_1, t, u) = 0} \qquad \boxed{q(L_1, t) = \frac{D_2 - D_1}{D_1}}$$

10

$$u(0, x) = f(x)$$

$$C(0, x) = C_0$$

$$u(0, x) = C_0$$

09 Parsial Differential Equation

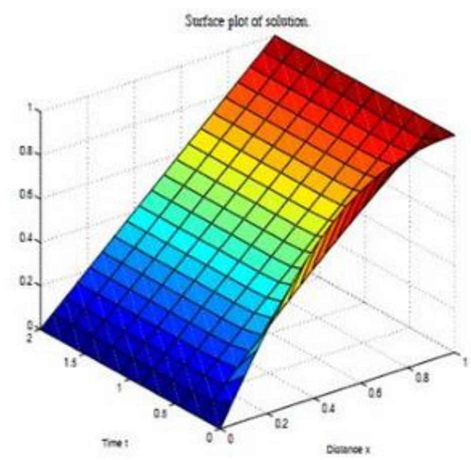
```
function [c,b,s] = system(x,t,u,DuDx)
    c = 1;
    b =D1*DuDx;
    s = 0;
end
```

```
function [pl,ql,pr,qr] =
 bc1(xl,ul,xr,ur,t)

pl = 0;
ql = 1/D1;
pr = 0;
qr = (D2-D1)/D1;
end
```

```
function value = initial1(x)

  value = C0;

end
```

```
m = 0;
%Define the solution mesh
x = linspace(0,1,20);
t = linspace(0,2,10);
%Solve the PDE
u = pdepe(m,@system,@initial1,@bc1,x,t);
%Plot solution
surf(x,t,u);
title('Surface plot of solution.');
xlabel('Distance x');
ylabel('Time t');
```

Results



Example 2.39: One-Dimensional Parabolic PDE

The temperature $u(x,t)$ in a wall of unit length can be described by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

The thickness of the wall is 1 m and the initial profile of the temperature in the wall at $t = 0$ sec is uniform at $T = 90^\circ$C. At time $t = 0$, the ambient temperature is suddenly changed to 15°C and held there. If we assume that there is no convection resistance, the temperature of both sides of the wall is also held constant at 15°C. Determine the temperature distribution graphically within the wall from $t = 0$ to $t = 21{,}600$ sec. The wall property can be assumed as $\alpha = 4.8 \times 10^{-7}$ m/sec^2.

Solution

Step 1: From the heat equation, we have $((1/\alpha)(\partial u/\partial t)) = (\partial^2 u/\partial x^2)$. Thus, we can see that $m = 0$, $g = 1/\alpha$, $f = \partial u/\partial x$, and $r = 0$. Create a function that holds g, f, and r:

```
function [g,f,r] = pdeTde(x,t,u,DuDx)
alpha = 4.8e-7;
g = 1/alpha;
f = DuDx;
r = 0;
end
```

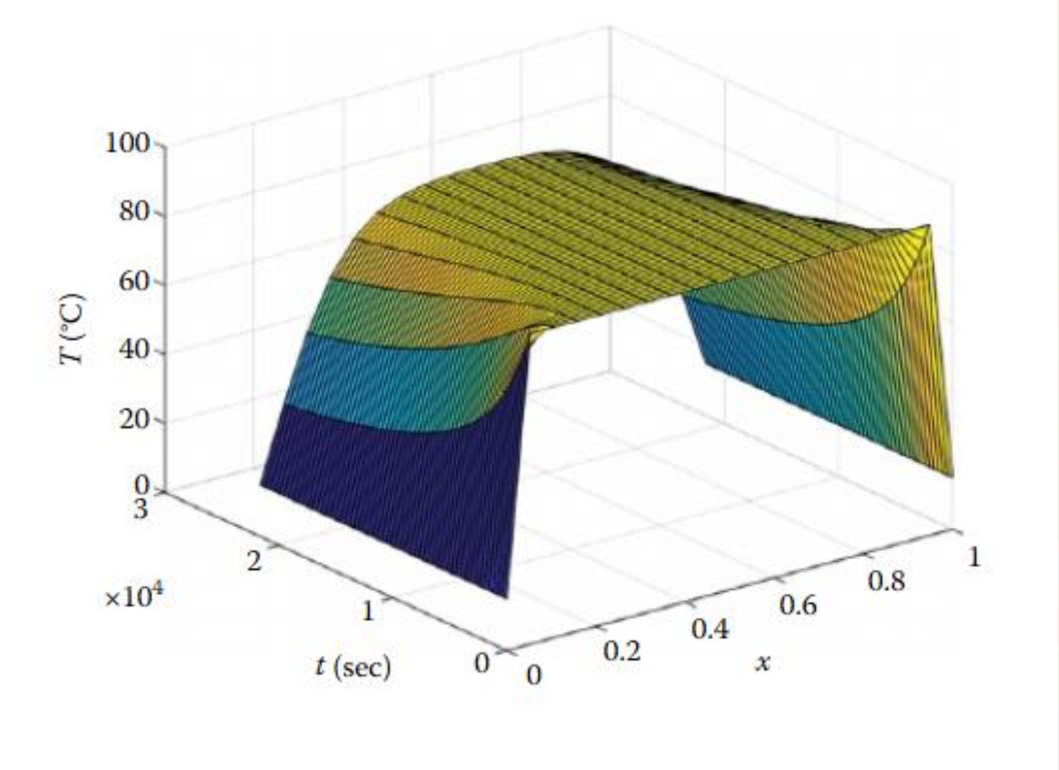
```
function u0 = pdeTic(x)
u0 = 90;
end
```

Step 3: Specify the boundary conditions $p(x, t, u) + q(x, t)f(x, t, u, (\partial u/\partial x)) = 0$. Since the temperature at both sides of the wall is 15°C, $p = pl = ul - 15$ and $q = ql = 0$ at $x = 0$, and $p = pr = ur - 15$ and $q = qr = 0$ at $x = 1$. Create a function that holds these boundary conditions.

```
function [pl,ql,pr,qr] = pdeTbc(xl,ul,xr,ur,t)
pl = ul-15;
ql = 0;
pr = ur-15;
qr = 0;
end
```

Step 4: Set the intervals for x and t, and call the function *pdepe* to solve the equation. Suppose that the range of x is divided into 20 subintervals and the time span is divided into 180 subintervals. The following script generates the temperature profile in the wall as shown in Figure 2.21:

```
>> m = 0; x = linspace(0,1,20);
>> t = linspace(0,21600,54);
```


100
80
60
40
20
0
T (°C)
3
2
1
0
×10^4
t (sec)
0
0.2
0.4
0.6
0.8
1
x

next

- MATLAB can perform symbolic calculus on expressions. Consider the following example:

syms x

f=sin(x^2)

f =

sin(x^2)

diff(f,x)

ans =

2*x*cos(x^2)

```
syms x y
q=x^2*y^3*exp(x)
q =
x^2*y^3*exp(x)
pretty(q)
  2  3
x  y  exp(x)
diff(q,y)
ans =
3*x^2*y^2*exp(x)
```

```
syms a t
u=exp(a*t)
u =
exp(a*t)
diff(u,t)
ans =
a*exp(a*t)
```

check whether the function $u(t)=e^{at}$ is a solution of the ODE

$$\frac{du}{dt} - au = 0.$$

syms a t

u=exp(a*t)

u =

exp(a*t)

diff(u,t)-a*u

ans =

0

$w(x,y)=\sin(\pi x)+\sin(\pi y)$ a solution of the differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0?$$

syms x y

w=sin(pi*x)+sin(pi*y)

w =

sin(pi*x) + sin(pi*y)

diff(w,x,2)+diff(w,y,2)

ans =

- pi^2*sin(pi*x) - pi^2*sin(pi*y)

simplify(ans)

ans =

-pi^2*(sin(pi*x) + sin(pi*y))

Since the result is not zero, the function w is not a solution of the PDE

To compute a mixed partial derivative, we have to iterate the **diff** command. Here is the mixed partial derivative of $w(x,y)=x^2+xy^2$ with respect to x and then y:

syms x y

w=x^2*exp(y)+x*y^2

w =

x^2*exp(y) + x*y^2

diff(diff(w,x),y)

ans =

2*y + 2*x*exp(y)

can use functions. Consider the following

```
clear
syms a x
f=@(x)exp(a*x)
f =
@(x)exp(a*x)
f(1)
diff(f(x),x)-a*f(x)
ans =
0
```

```
syms x
f=x^2
f =
x^2
int(f,x)
ans =
x^3/3
```

To compute a definite integral

int(x^2,x,0,1)

ans =

1/3

f=@(x)exp(cos(x))

f =

@(x)exp(cos(x))

quad(f,0,1)

ans =

2.3416

"quad" is short for quadrature, another term for numerical integration

$$\frac{d}{dx}\int_c^d F(x,y)\,dy = \int_c^d \frac{\partial F}{\partial x}(x,y)\,dy.$$

```
syms x y c d
f=x*y^3+x^2*y
f =
x^2*y + x*y^3
r1=diff(int(f,y,c,d),x)
r1 =
- (x*(c^2 - d^2))/2 - ((c^2 - d^2)*(c^2 + d^2 + 2*x))/4
r2=int(diff(f,x),y,c,d)
r2 =
-((c^2 - d^2)*(c^2 + d^2 + 4*x))/4
r1-r2
ans =
((c^2 - d^2)*(c^2 + d^2 + 4*x))/4 - ((c^2 - d^2)*(c^2 + d^2 + 2*x))/4 -
(x*(c^2 - d^2))/2
simplify(ans)
ans =
0
```

The MATLAB solve command

- to solve the linear equation $ax+b=0$ for x.

syms f x a b

f=a*x+b

f =

b + a*x

solve(f,x)

ans =

-b/a

To solve x^2-3*x+2=0

```
syms f x
f=x^2-3*x+2;
solve(f,x)
ans =
1
2
```

to solve the equations

x+y=1

2x-y=1

```
syms x y
s=solve(x+y-1,2*x-y-1,x,y)
s.x
ans =
2/3
s.y
ans =
1/3
```

s=solve(x^2+y^2-1,y-x^2,x,y)

s =

x: [4x1 sym]

y: [4x1 sym]

The first solution is

pretty(s.x(1))

pretty(s.y(1))

The second solution is

pretty(s.x(2))

pretty(s.y(2))

```
clear
syms u1 u2 u3
u1=sym([1;0;2]);
u2=sym([0;1;1]);
u3=sym([1;2;-1]);
A=[u1,u2,u3]

A =
[ 1, 0, 1]
[ 0, 1, 2]
[ 2, 1, -1]

b=sym([8;2;-4]);
x=A\b
x =
18/5
-34/5
22/5
```