Deret MacLaurin

Deret Taylor

Tujuan

- Kenapa perlu perkiraan?
  - Perkiraan dibentuk dari fungsi paling sederhana – polynomial.
  - Kita bisa mengintegrasikan dan mendiferensiasi dengan mudah.
  - Kita bisa gunakan saat kita tidak tahu fungsi sebenarnya.

Polynomial Approximations

- Misalkan kita ingin membuat perkiraan untuk sebuah fungsi yang kompleks pada sekitar $x = 0$;
- Perkiraan paling simple adalah menentukan sebuah konstanta, sehingga:

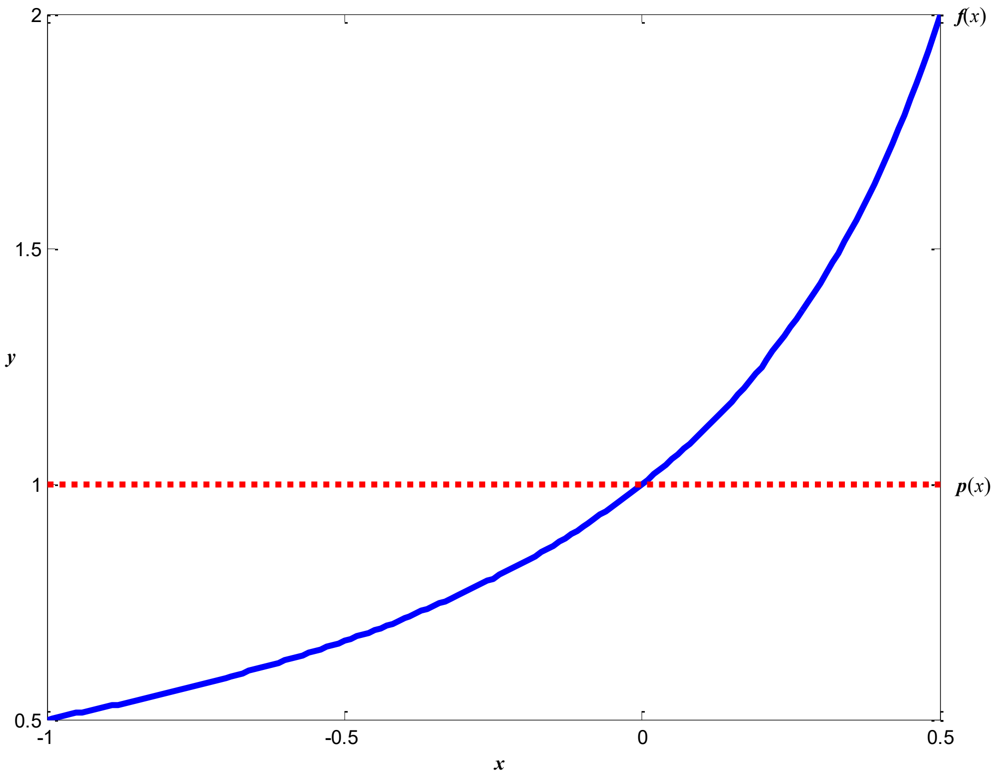
$$p_0(x) = a_0$$

- Catatan: perkiraan di atas disebut sebagai zero'th order polynomial approximation;
- Lalu, nilai berapa yang harus kita berikan pada konstanta itu?

Polynomial Approximations

- Kita inginkan angka paling akurat pada $x = 0$.
- Sehingga:

$$p_0(x) = f(0)$$



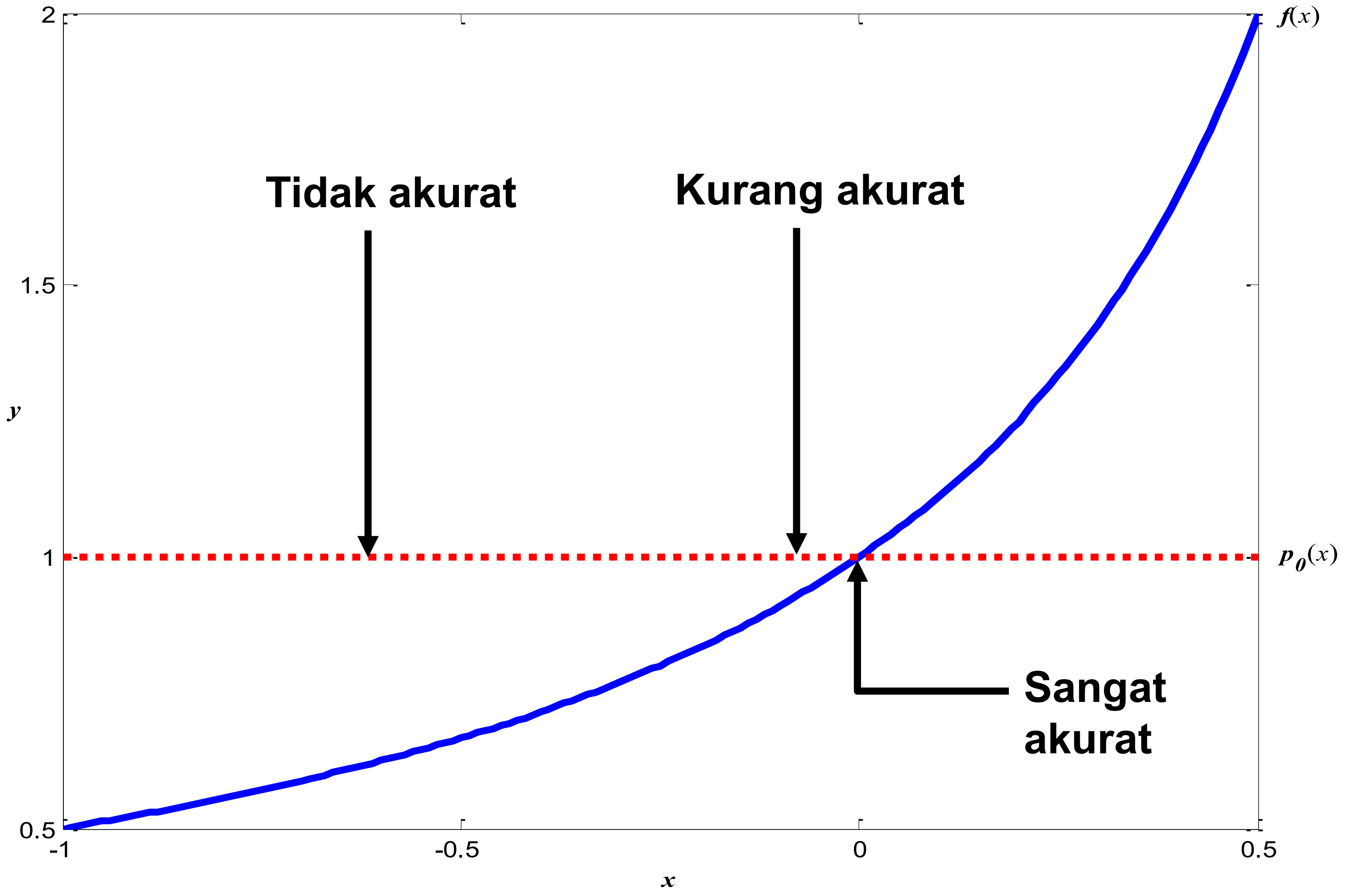
Polynomial Approximations

- Contoh

$$f(x) = \frac{1}{1-x}$$

$$f(0) = \frac{1}{1} = 1 \Rightarrow p_0(x) = 1$$

Polynomial Approximations



Polynomial Approximations

- Sekarang kita tingkatkan dengan perkiraan dengan menggunakan aproksimasi linier (1st order approximation);

$$p_1(x) = a_0 + a_1 x$$

- Sekarang kita pilih nilai sehingga perpotongan dan garis nya semirip mungkin dengan fungsi sebenarnya.

Polynomial Approximations

- Menyamakan perpotongan:

$$p_1(0) = f(0) \Rightarrow a_0 + a_1 \times 0 = f(0)$$
$$\Rightarrow a_0 = f(0)$$

- Menyamakan slope:

$$p_1'(0) = f'(0) \Rightarrow a_1 = f'(0)$$

- Sehingga polinom nya:

$$p_1(0) = f(0) + f'(0)x$$

Polynomial Approximations

- Contoh

$$f(x) = \frac{1}{1-x}$$

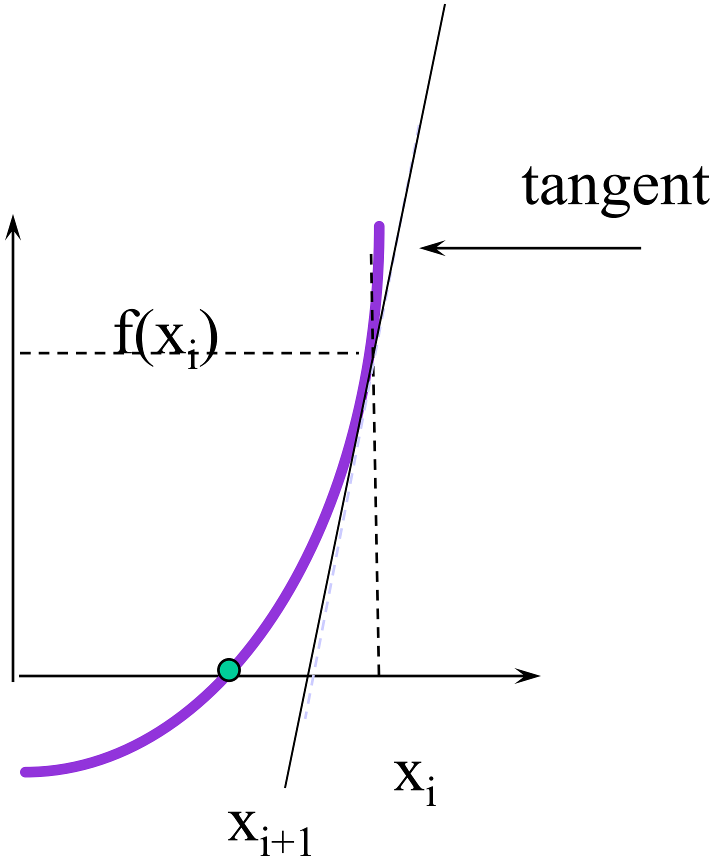
$$p_1(x) = a_0 + a_1 x$$

$$f(0) = \frac{1}{1-0} = 1 \Rightarrow a_0 = f(0) = 1$$

$$f'(0) = \frac{1}{(1-x)^2} = 1 \Rightarrow a_1 = f'(0) = 1$$

$$\Rightarrow p_1(x) = 1 + x$$

Ingat, Metode Newton Raphson



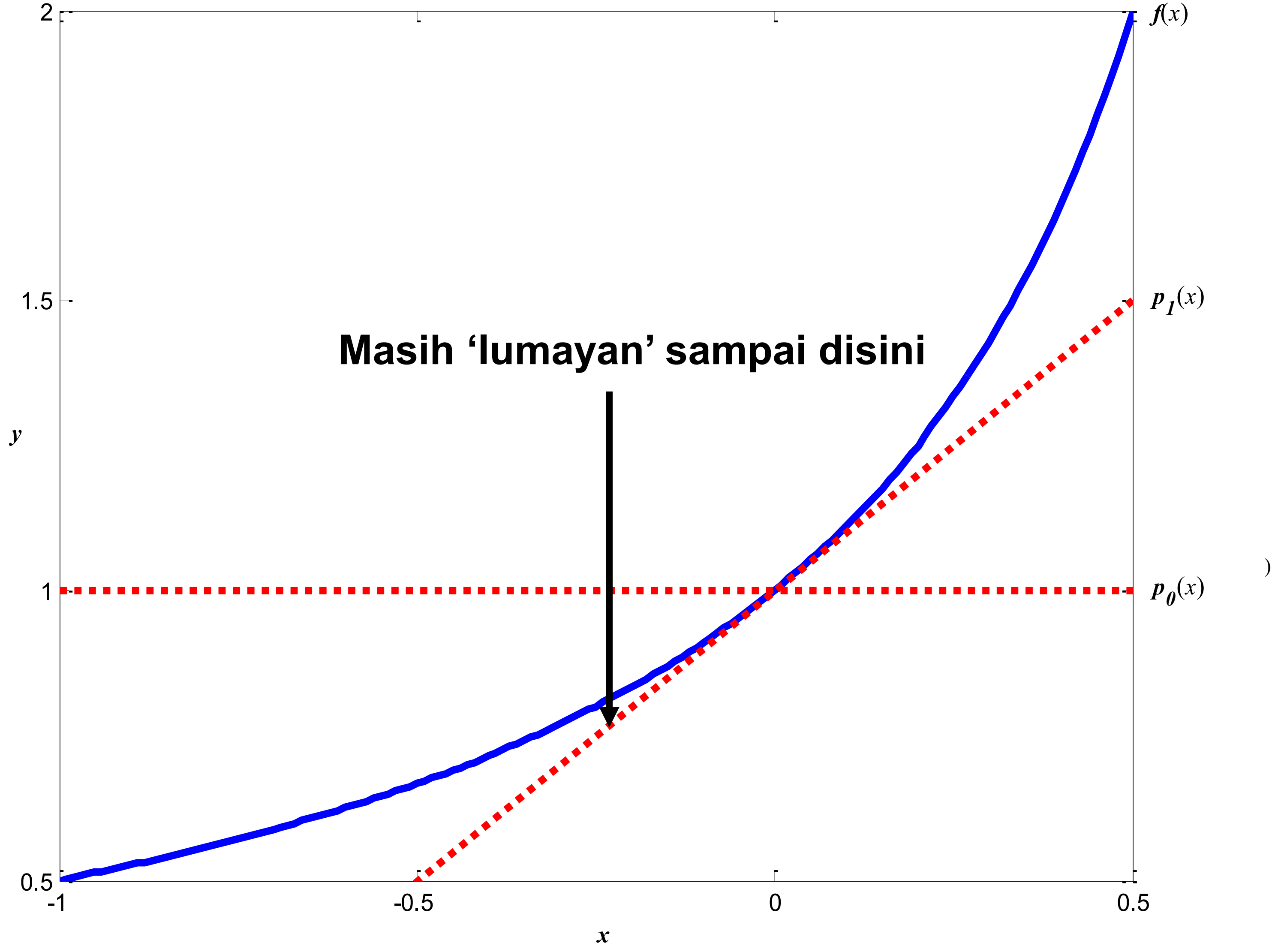
$$tangent = \frac{dy}{dx} = f'$$

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

rearrange

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Polynomial Approximations



Polynomial Approximations

- Sekarang coba dengan perkiraan kuadratik:

$$p_2(x) = a_0 + a_1 x + a_2 x^2$$

- Kita inginkan perpotongan, gradient dan kurva (turunan kedua) dari perkiraan kita dapat *match* dengan fungsi sebenarnya pada $x = 0$.

Polynomial Approximations

- Menyamakan perpotongan:

$$p_2(0) = f(0) \Rightarrow a_0 + a_1 \times 0 + a_2 \times 0^2 = f(0)$$
$$\Rightarrow a_0 = f(0)$$

- Menyamakan kemiringan:

$$p_2'(0) = f'(0) \Rightarrow a_1 + 2a_2 \times 0 = f'(0)$$

Polynomial Approximations

- Mencocokkan kurva (turunan ke 2):

$$p_2''(0) = f''(0) \quad \Rightarrow 2a_2 = f''(0)$$

$$\Rightarrow a_2 = \frac{1}{2} f''(0)$$

- Memberikan polinom

$$p_2(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2$$

Polynomial Approximations

- Contoh

$$f(x) = \frac{1}{1-x}$$

$$p_2(x) = a_0 + a_1 x + a_2 x^2$$

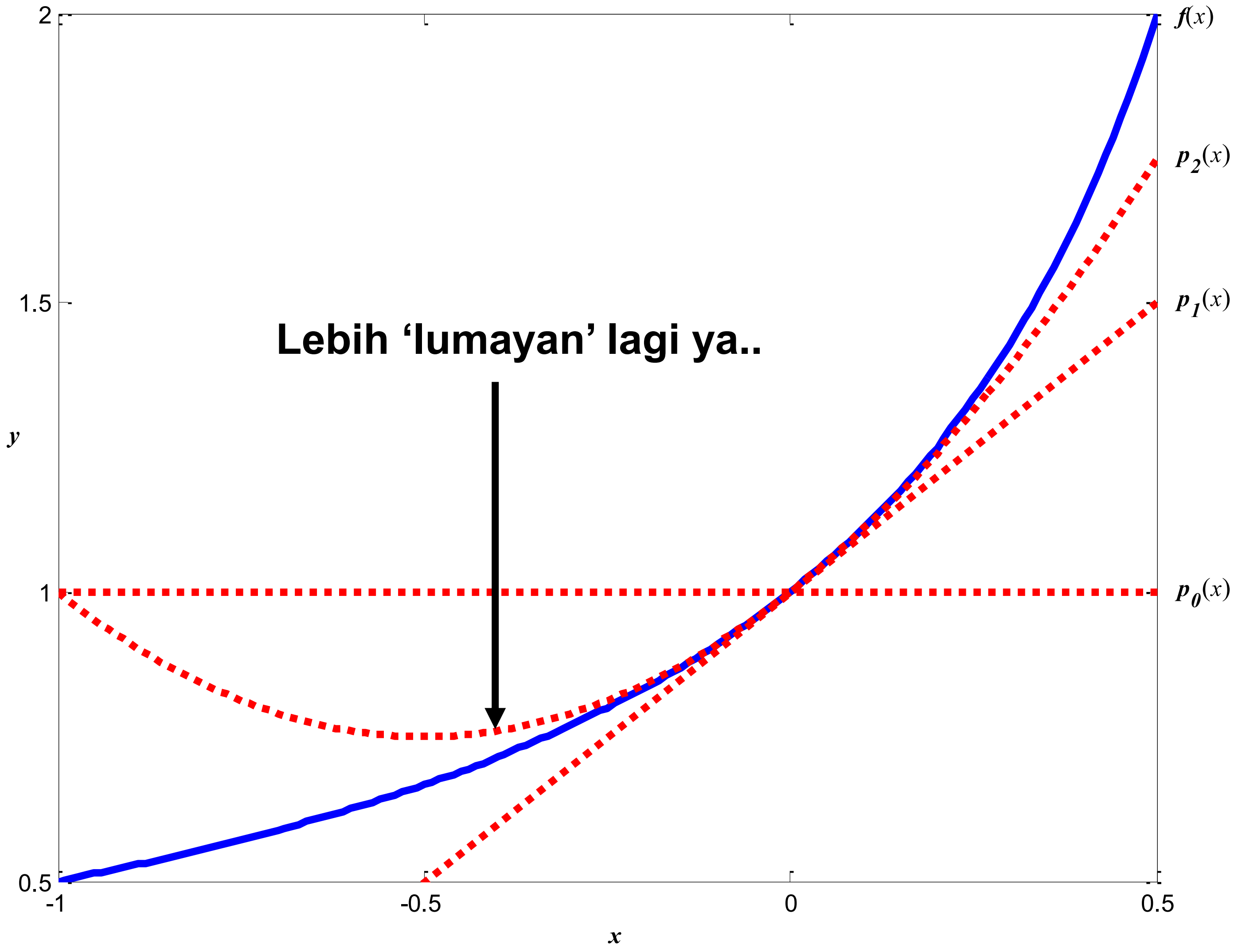
- Dari sebelumnya: $a_0 = 1, a_1 = 1$

$$f''(x) = \frac{2}{(1-x)^3} \Rightarrow 2a_2 = f'(0) = 2$$

$$\Rightarrow a_2 = 1$$

$$\Rightarrow p_2(x) = 1 + x + x^2$$

Polynomial Approximations



Polynomial Approximations

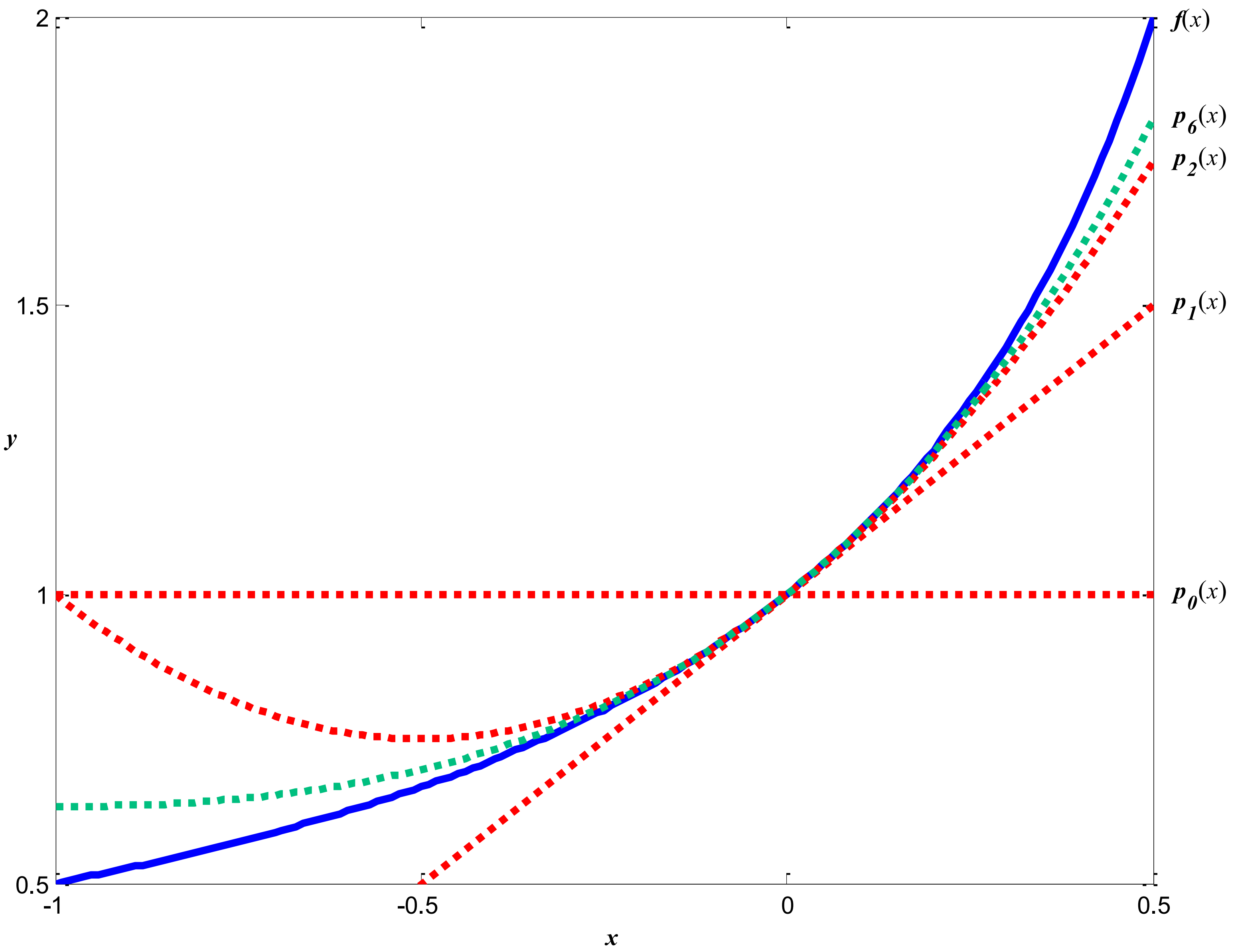
- Kita bisa teruskan penaksiran secara polinom hingga *n* derajad.
- Kalau kita teruskan, kita akan mendapatkan rumus:

$$f(x) \approx p_n(x) = f(0) + f'(0)x$$

$$+ f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!}$$

Polynomial Approximations

- Akurasi perkiraan akan bertambah seiring dengan penambahan polinom;
- Kita lihat polinom derajad 0, 1, 2 dan 6 (warna hijau), dibanding fungsi asli nya f(x) (warna biru).



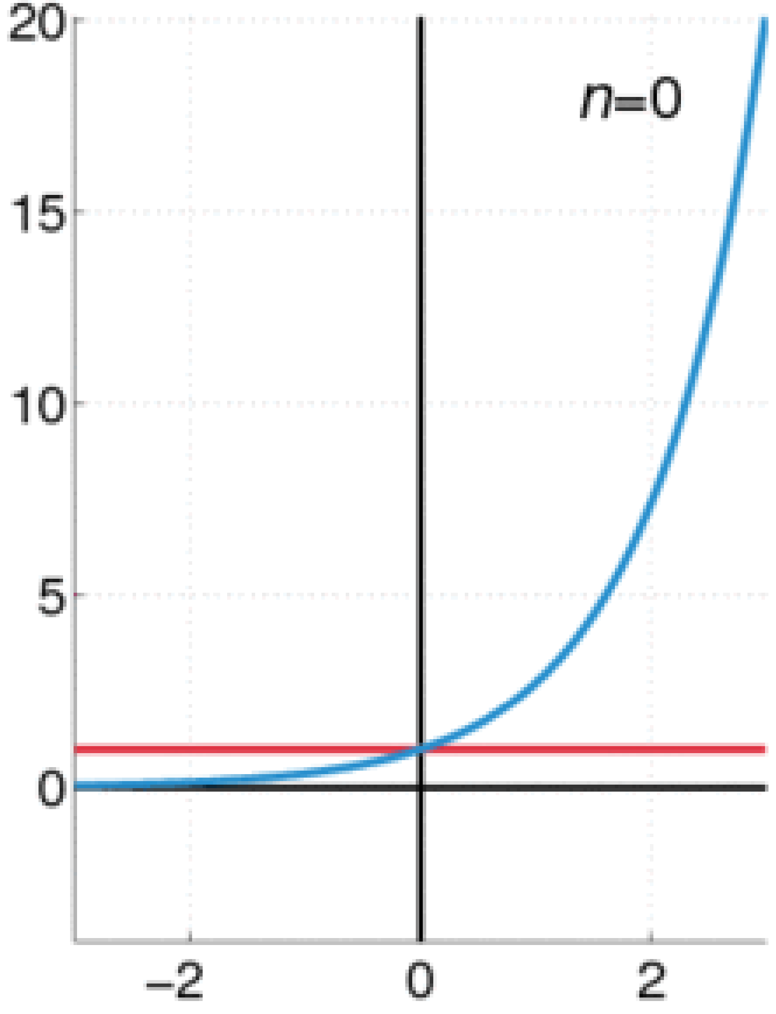
2
1.5
1
0.5
-1
-0.5
0
0.5
y
x
f(x)
p6(x)
p2(x)
p1(x)
p0(x)

Maclaurin (Power) Series

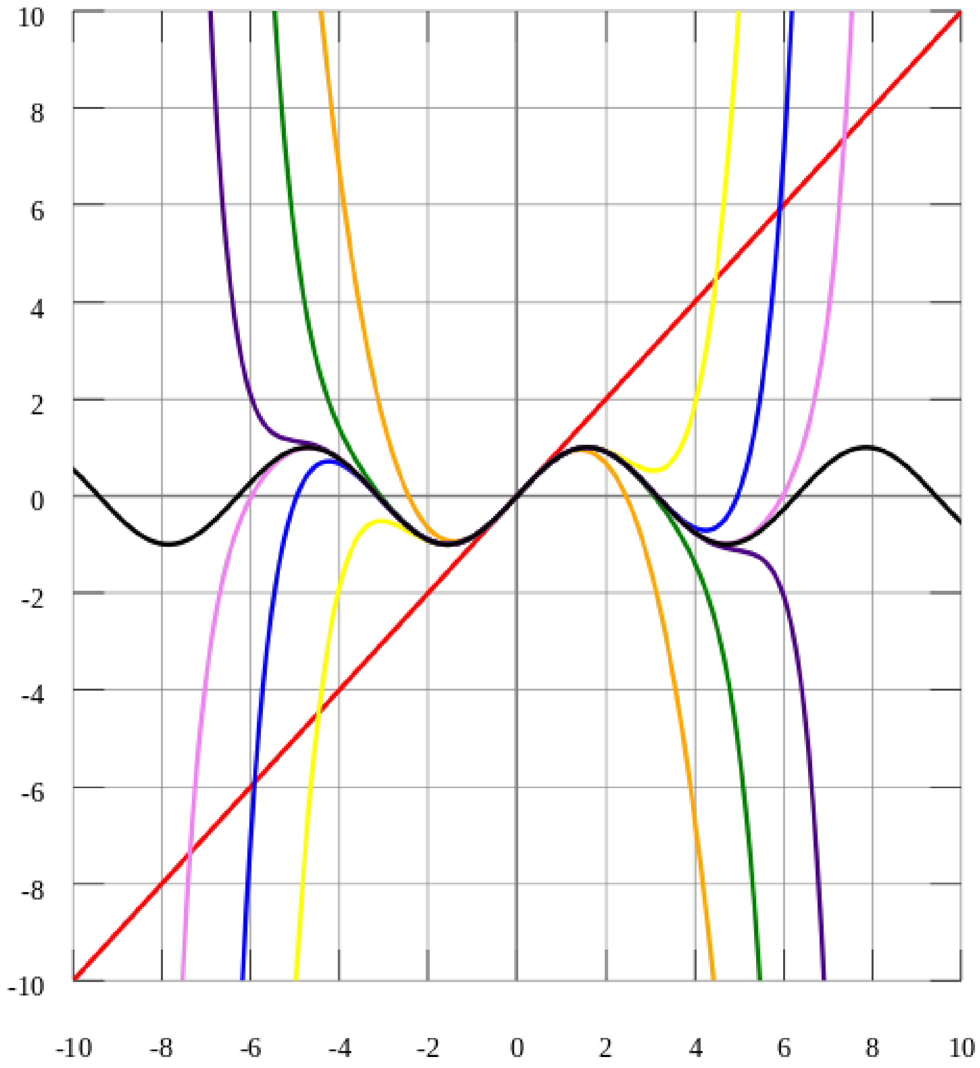
- Deret Maclaurin adalah penaksiran polinom derajad tak hingga

$$f(x) = f(0) + f'(0)x$$
$$+ f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!} + \ldots$$

- Notice: Deret infinite (tak hingga) menyatakan bahwa akhirnya deret ini sama dengan fungsi sebenarnya, bukan penaksiran lagi!



20
15
10
5
0
−2
0
2
n=0



10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10

Taylor Series

- Dari awal kita selalu memulai perkiraan pada nilai $x = 0$
- Sesungguhnya, kita bisa membuat deret polinom yang berasal dari titik manapun. $x = x_0$
- Ini disebut ***Taylor Series.***
- **Jadi, Deret MacLaurin** merupakan Deret Taylor yang berpusat pada x_0=0

Taylor Series

- Rumus umum Deret Taylor:

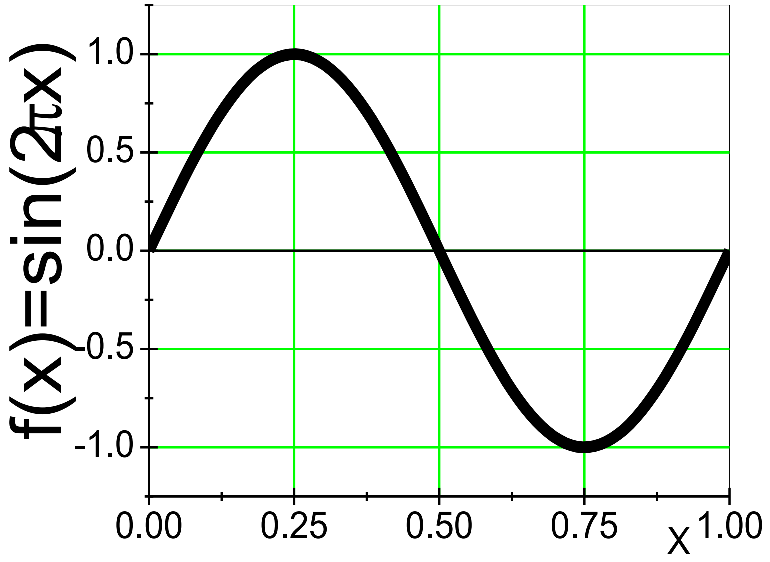
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$+ \ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + \ldots$$

$$= \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x - x_0)^n}{n!}$$

Taylor Series

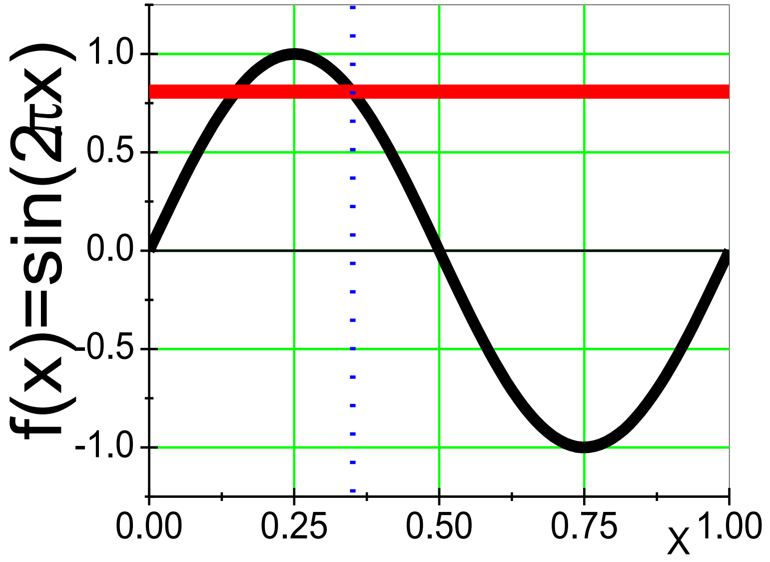
- Approximate function? Copy derivatives!



What is f(x) near x=0.35?

Taylor Series

- Approximate function? Copy derivatives!

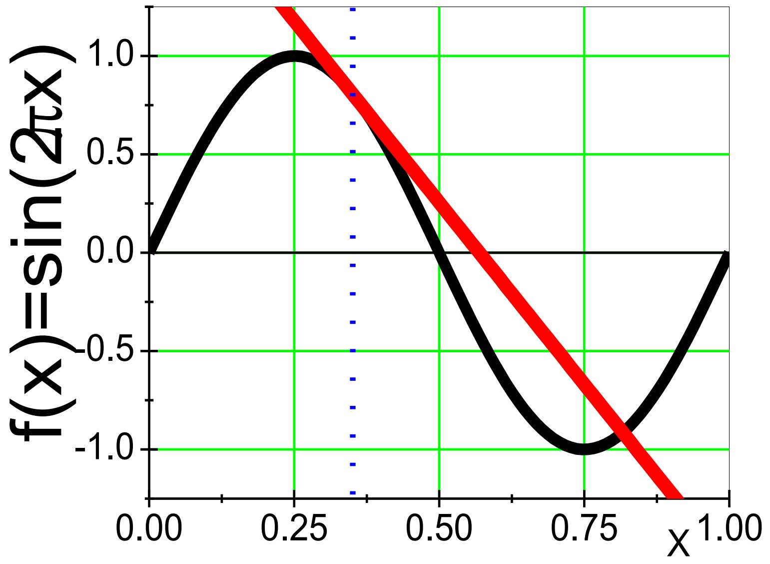


What is f(x) near x=0.35?

$$T_0(x) = f(0.35)$$

Taylor Series

- Approximate function? Copy derivatives!

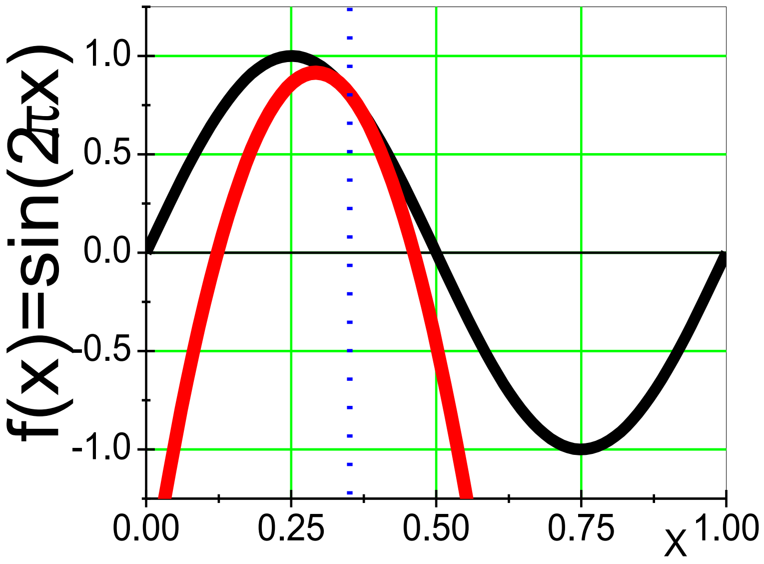


What is f(x) near x=0.35?

$$T_1(x) = f(0.35) + f'(0.35)\left(x - 0.35\right)$$

Taylor Series

- Approximate function? Copy derivatives!

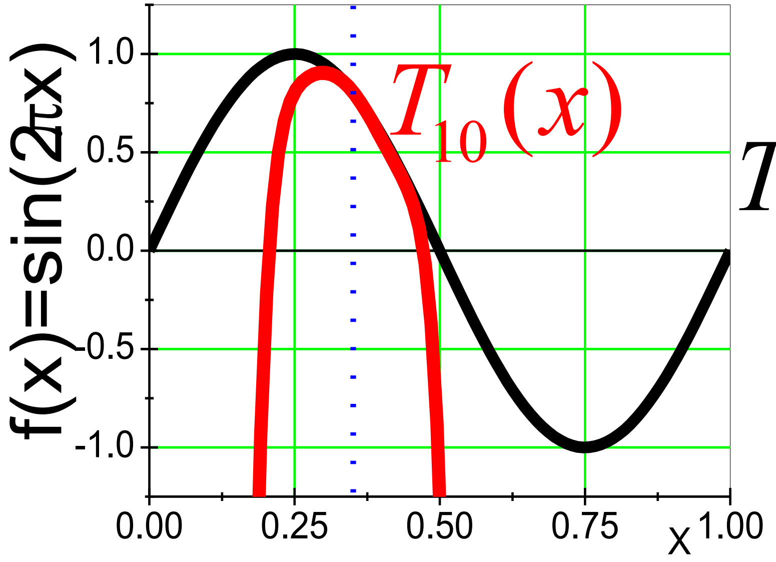


What is f(x) near x=0.35?

$$T_2(x) = f(0.35) + f'(0.35)(x - 0.35) + \tfrac{1}{2} f''(0.35)(x - 0.35)^2$$

Taylor Series

- Approximate function? Copy derivatives!



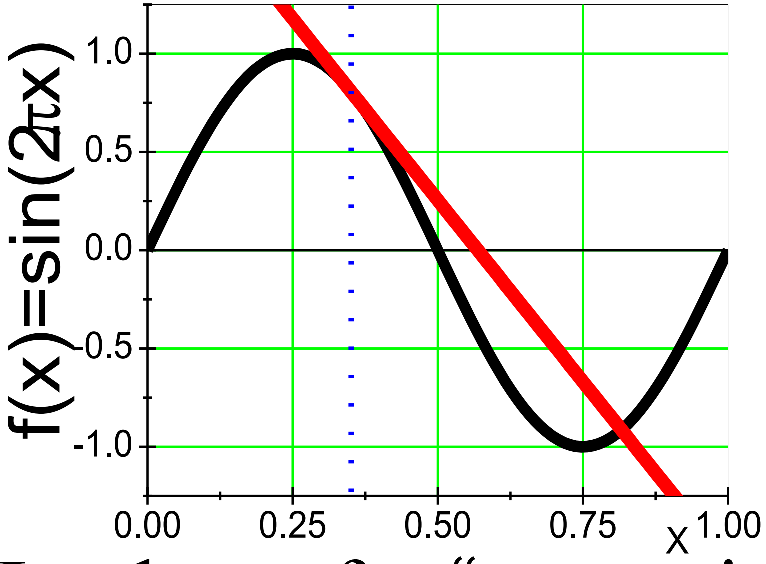
What is f(x) near x=0.35?

$$T_2(x) = f(0.35) + f'(0.35)(x-0.35) + \frac{1}{2} f''(0.35)(x-0.35)^2 \ldots$$

$$T_N(x) = \sum_{i=0}^{N} \frac{f^{(i)}(a)(x-a)^i}{i!}$$

Taylor Series

- Approximate function? Copy derivatives!



Most Common: 1st Order

$$T_1(x) = f(a) + f'(a)(x - a)$$

- Look out for "approximate" or "when x is small" or "small angle" or "close to" …

Contoh – Taylor Series

- Bentuklah Deret Taylor untuk:

$$f(x) = \ln(x), \qquad x_0 = 1$$

- Cari nilai fungsi dan turunannya untuk fungsi pada x_0=1

Contoh – Taylor Series

$$f(x) = \ln(x) \quad \Rightarrow f(x_0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow f'(x_0) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow f''(x_0) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow f'''(x_0) = \frac{2}{1^3} = 2$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{x^n}$$

$$\Rightarrow f^{(n)}(x_0) = \frac{(n-1)!(-1)^{n-1}}{1^n} = (n-1)!(-1)^{n-1}$$

Contoh – Taylor Series

- Gunakan Rumus Umum Deret Taylor:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$+ \ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = 0 + (x - 1) - \frac{(x-1)^2}{2!} + \frac{2!(x-1)^3}{3!}$$

$$+ \ldots + (n-1)!(-1)^{n-1}\frac{(x-1)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$+ \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + \ldots$$

Truncated Taylor Series

- We cannot evaluate a Taylor series – it is infinite!
- Kita bisa memutuskan untuk membuat perkiraan dari sebuah fungsi hingga n (derajat) tertentu yang *tidak* tak terhingga;
- Kita sebut sebagai ***Truncated Taylor Series.***

Truncated Taylor Series

- To find an *n*th order truncated Taylor series

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$+ \ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!}$$

- **<u>Note</u>**: This is the same concept as the polynomial approximations we introduced earlier.

Example – Truncated Taylor Series

- Find a cubic (degree 3) truncated Taylor series for the function:

$$f(x) = \cos(2x)$$

centered at:

$$x = \frac{\pi}{4}$$

Example – Truncated Taylor Series

- For a degree 3 approximation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + f'''(x_0)\frac{(x - x_0)^3}{3!}$$

- So we need to evaluate the function and its first three derivatives at the center.

Example – Truncated Taylor Series

- Evaluating these:

$$f(x) = \cos(2x) \qquad \Rightarrow f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = -2\sin(2x) \qquad \Rightarrow f'\left(\frac{\pi}{4}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2$$

$$f''(x) = -4\cos(2x) \qquad \Rightarrow f''\left(\frac{\pi}{4}\right) = -4\cos\left(\frac{\pi}{2}\right) = 0$$

$$f'''(x) = 8\sin(2x) \qquad \Rightarrow f'''\left(\frac{\pi}{4}\right) = 8\sin\left(\frac{\pi}{2}\right) = 8$$

Example – Truncated Taylor Series

- … which gives:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

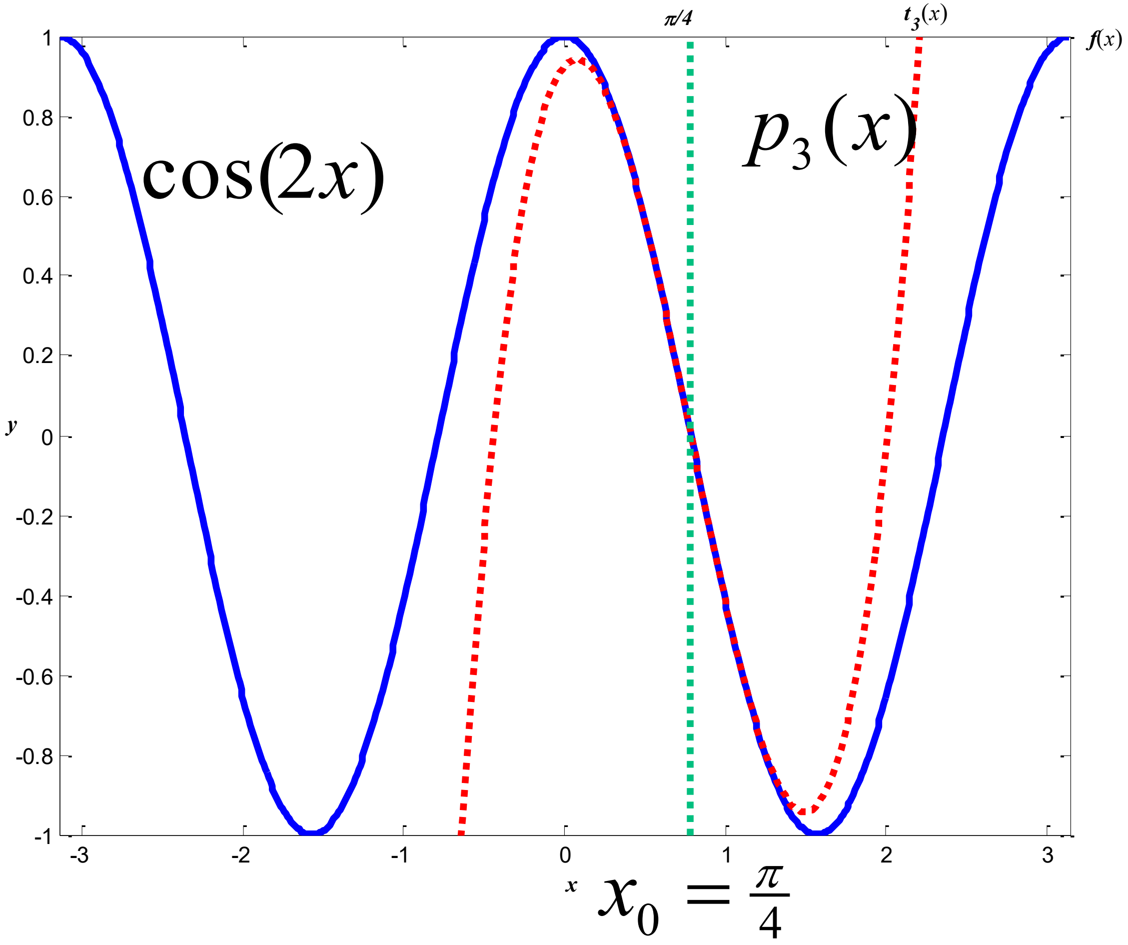
$$+ f''(x_0)\frac{(x - x_0)^2}{2!} + f'''(x_0)\frac{(x - x_0)^3}{3!}$$

$$\Rightarrow f(x) \approx 0 - 2 \times \left(x - \frac{\pi}{4}\right)$$

$$+ 0 \times \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} + 8 \times \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}$$

$$\Rightarrow f(x) \approx -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3$$

Example – Truncated Taylor Series



Series Accuracy

- Kenapa mesti pakai Deret Taylor kalau bisa pakai Maclaurin?
- Perkiraan kita akan makin jauh dari akurat jika semakin jauh dari titik awal x_0;
- Kita harus selalu memakai titik awal yang dekat dengan titik yang akan diperkirakan dan juga mudah untuk melakukan perkiraan.