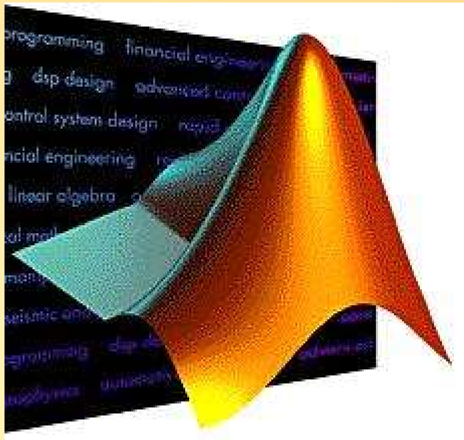
09 Parsial Differential Equation

Parsial Differential Equation



Persamaan differensial parsial secara umum untuk orde dua

Persamaan differensial parsial secara umum untuk orde dua dalam variabel bebas x dan y dapat dinyatakan sebagai berikut :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Persamaan differensial parsial dapat diklasifikasikan tergantung dari nilai $B^2 - 4AC$.

- jika $B^2 - 4AC < 0$, maka persamaan Eliptik
- jika $B^2 - 4AC = 0$, maka persamaan Parabolik
- jika $B^2 - 4AC > 0$, maka persamaan Hiperbolik

- Jika koefisien A, B, dan C adalah fungsi x, y, dan/atau u, persamaan mungkin berubah dari satu klasifikasi menjadi klasifikasi lain pada titik bervariasi.
- Dalam teknik kimia persamaan yang sering dijumpai adalah persamaan differensial eliptik dan parabolik, sehingga kedua persamaan itulah yang akan dibahas dalam kuliah ini.

PERSAMAAN DIFFERENSIAL ELIPTIK

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Persamaan differensial eliptik terbentuk jika koefisien A dan C pada persamaan umum sama dengan 1 dan B sama dengan nol, sehingga $B^2 - 4AC < 1$.

Ada 2 type persamaan differensial eliptik yang akan dibahas, yaitu

- Persamaan Laplace

$$A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} = 0$$

- Persamaan Poisson

$$A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

PERSAMAAN LAPLACE

- Persamaan Laplace sering muncul dari penyusunan persoalan perpindahan panas dalam suatu plat.
- Bentuk paling sederhana persamaan Laplace adalah

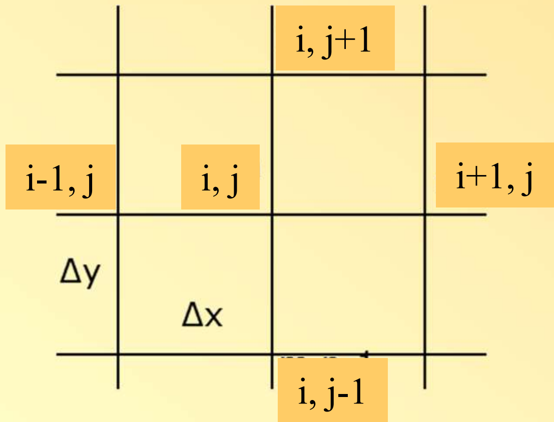
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Penyelesaian persamaan Laplace adalah metode beda hingga.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u_{i+1,j} - 2u_{i,j} +_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} +_{i,j-1}}{(\Delta y)^2}$$

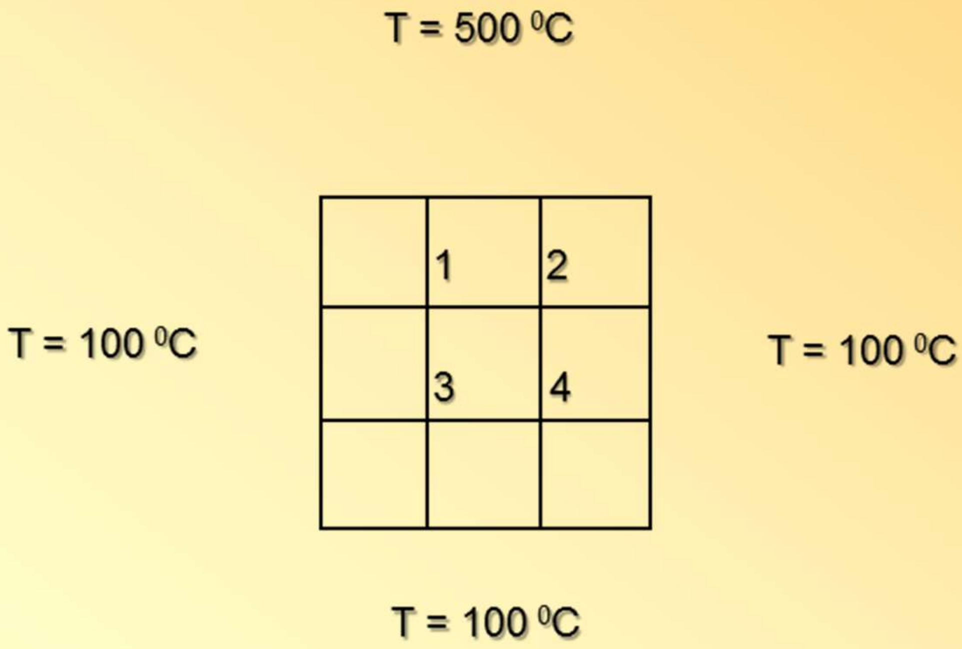
Jika diambil $\Delta x = \Delta y = h$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\,[\,u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}] = 0$$

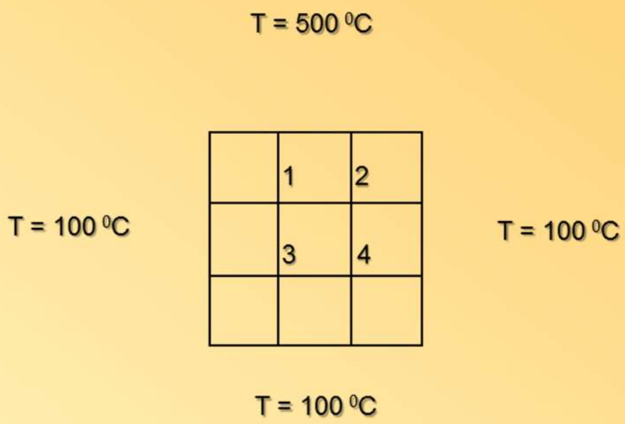


Contoh 1

Plat tembaga tipis dengan ukuran 3 cm x 3 cm. Permukaan salah satu sisi dipertahankan 500 ^{o}C dan ketiga sisi yang lain dipertahankan pada suhu 100 ^{o}C. Permukaan plat diisolasi sehingga panas mengalir arah x dan y saja. Tentukan distribusi suhu plat tersebut pada keadaan tunak (*steady)*.



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \right] = 0$$



$T2 + 100 + 500 + T3 - 4T1 = 0$

$100 + T1 + 500 + T4 - 4T2 = 0$

$T4 + 100 + T1 + 100 - 4T3 = 0$

$100 + T3 + T2 + 100 - 4T4 = 0$

$$\begin{aligned}
-4T1 + T2 + T3 \quad &= -600 \\
T1 - 4T2 \quad + T4 &= -600 \\
T1 \quad - 4T3 + T4 &= -200 \\
T2 + T3 - 4T4 &= -200
\end{aligned}$$

$$T = \begin{matrix} 250 \\ 250 \\ 150 \\ 150 \end{matrix}$$

Contoh 2

Plat tembaga tipis dengan ukuran 6 cm x 8 cm. Permukaan salah satu sisi dengan panjang 6 cm dipertahankan 100 ^oC dan ketiga sisi yang lain dipertahankan pada suhu 40 ^oC. Permukaan plat diisolasi sehingga panas mengalir arah x dan y saja. Tentukan distribusi suhu plat tersebut pada keadaan tunak (steady).

Contoh 3

Plat tipis dari baja mempunyai ukuran 10 cm x 20 cm. Jika salah satu sisi ukuran 10 cm dijaga pada 100 ^{O}C dan ketiga sisi yang lain dijaga pada 0 ^{O}C. Tentukan profil temperatur pada plat. Untuk baja k = 0,16 kal/detik.cm^2.C/cm.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

dengan $u(x,0) = 0,$

$u(x,10) = 0,$

$u(0,y) = 0,$

$u(20,y) = 100.$

```
clear;
N=20;
M=10;

for j=0:N-2
  for i=1:M-2
     X(i+j*(M-1))=0;
  end
end
for i=1:M-1
   X(i*(N-1))=-100;
end
```

```
% Penyusunan matriks A
for i=1:(N-1)*(M-1)
   A(i,i)=-4;
end
for i=1:N-2
   for k=0:M-2
   A(i+k*(N-1),i+1+k*(N-1))=1;
   end
end
for k=0:M-3
   for i=1:N-1
   A(i+k*(N-1),i+(k+1)*(N-1))=1;
   end
end
for i=1:(N-1)*(M-1)
   for j=1:i
   A(i,j)=A(j,i);
   end
end
```

```
% Inversi Matriks dan Perhitungan Temperatur
G=inv(A);
U=G*X';

% Plot hasil bentuk contour
for i=1:M-1
   for j=1:N-1
   x(i,j)=U(j+(i-1)*(N-1));
   end
end
T = x
[i,j]=meshgrid(1:1:N-1,1:1:M-1);
[c,h]=contourf(i,j,x);
```

PERSAMAAN PARABOLIK

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{k}\frac{\partial u}{\partial t}$$

METODE EKSPLISIT

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{(\Delta x)^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^{j}}{\Delta t}$$

$$u_i^{j+1} = \frac{k\Delta t}{c\rho(\Delta x)^2}\left(u_{i+1}^j + u_{i-1}^j\right) + \left(1 - \frac{2k\Delta t}{c\rho(\Delta x)^2}\right)u_i^j$$

Penyederhanaan $\frac{k\Delta t}{c\rho(\Delta x)^2} = \frac{1}{2} = \frac{1}{M}$

$$u_i^{j+1} = \frac{1}{2}\left(u_{i+1}^j + u_{i-1}^j\right)$$

Distribusi Temperatur sebagai Fungsi Waktu pada Plat Tipis

Plat besi yang sangat luas mempunyai tebal 2 cm. Temperatur mula-mula dalam plat merupakan fungsi jarak dari salah satu sisinya sebagai berikut :

$u = 100x$ untuk $0 < x < 1$,

$u = 100(2 - x)$ untuk $1 < x < 2$.

Tentukan temperatur tebal plat sebagai fungsi x dan t, jika kedua permukaan tetap dijaga 0 ^{O}C. Untuk besi k = 0,13 kal/detik.cm. ^{O}C, c = 0,11 cal/g. ^{O}C, ρ = 7,8 g/cm^3.

$$\frac{k\Delta t}{c\rho(\Delta x)^2} = \frac{1}{2} = \frac{1}{M}$$

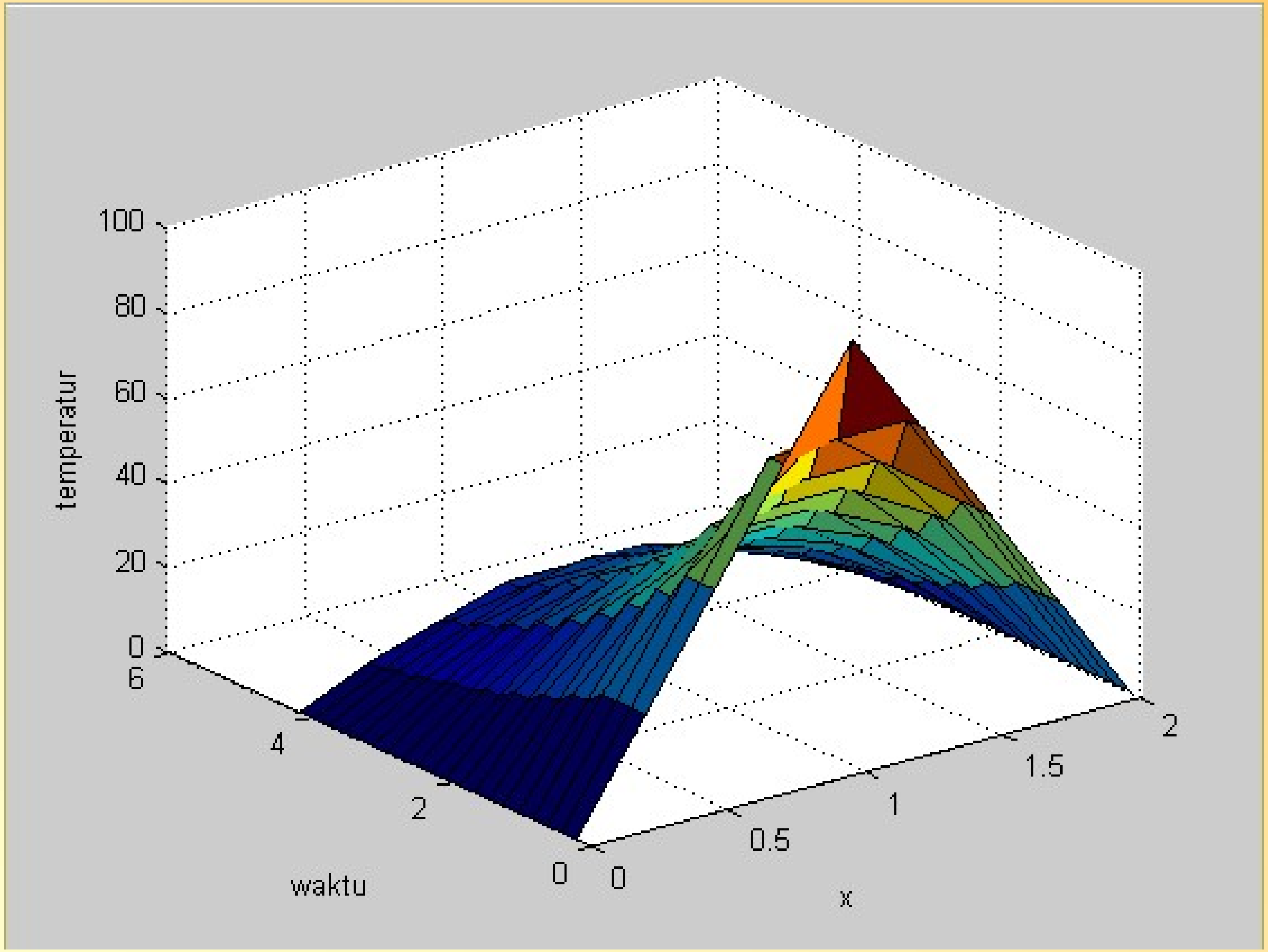
$\Delta x = 0{,}25$ sehingga $\Delta t = 0{,}206$ detik

time	x = 0	x = 0.25	x = 0.5	x = 0.75	x = 1	x = 1.25	x = 1.5	x = 1.75	x = 2
0	0	25.00	50.00	75.00	100.00	75.00	50.00	25.00	0
0.206	0	25.00	50.00	75.00	75.00	75.00	50.00	25.00	0
0.412	0	25.00	50.00	62.50	75.00	62.50	50.00	25.00	0
0.618	0	25.00	43.75	62.50	62.50	62.50	43.75	25.00	0
0.824	0	21.88	43.75	53.13	62.50	53.13	43.75	21.88	0
1.03	0	21.88	37.50	53.13	53.13	53.13	37.50	21.88	0
1.236	0	18.75	37.50	45.31	53.13	45.31	37.50	18.75	0
1.442	0	18.75	32.03	45.31	45.31	45.31	32.03	18.75	0
1.648	0	16.02	32.03	38.67	45.31	38.67	32.03	16.02	0
1.854	0	16.02	27.34	38.67	38.67	38.67	27.34	16.02	0
2.06	0	13.67	27.34	33.01	38.67	33.01	27.34	13.67	0
2.266	0	13.67	23.34	33.01	33.01	33.01	23.34	13.67	0
2.472	0	11.67	23.34	28.17	33.01	28.17	23.34	11.67	0
2.678	0	11.67	19.92	28.17	28.17	28.17	19.92	11.67	0
2.884	0	9.96	19.92	24.05	28.17	24.05	19.92	9.96	0
3.09	0	9.96	17.00	24.05	24.05	24.05	17.00	9.96	0
3.296	0	8.50	17.00	20.53	24.05	20.53	17.00	8.50	0
3.502	0	8.50	14.51	20.53	20.53	20.53	14.51	8.50	0
3.708	0	7.26	14.51	17.52	20.53	17.52	14.51	7.26	0
3.914	0	7.26	12.39	17.52	17.52	17.52	12.39	7.26	0
4.12	0	6.19	12.39	14.95	17.52	14.95	12.39	6.19	0
4.326	0	6.19	10.57	14.95	14.95	14.95	10.57	6.19	0
4.532	0	5.29	10.57	12.76	14.95	12.76	10.57	5.29	0

```
clc
clear all

% Data-data
L=2;
k=0.13;
c=0.11;
rho=7.8;

% Interval
N=8;
M=0.5;
delx=L/N;
delt=M*c*rho*delx^2/k;
xo=0;
Jend=20;

for i=1:N+1
   x(i)=xo+delx*(i-1);
end

% Kondisi awal
for i=1:ceil((N+1)/2)
   u(1,i)=100*x(i);
end
for i=N+1:-1:ceil((N+1)/2)
   u(1,i)=100*(2-x(i));
end
for i=1:Jend
   t(i)=delt*i;
end
```

```
for j=1:Jend
    for i=2:ceil((N+1)/2)+1
        u(j+1,i)=(k*delt/(c*rho*delx^2))*(u(j,i-1)+u(j,i+1))+...
            (1-2*k*delt/(c*rho*delx^2))*u(j,i);
    end
    for i=N+1:-1:ceil((N+1)/2)+1
        u(j+1,i)=u(j+1,ceil((N+1)/2)*2-i);
    end
end
t', u
surf(x, t', u(1:Jend,:))
xlabel('x');  ylabel('waktu');
zlabel(‘konsentrasi')
```

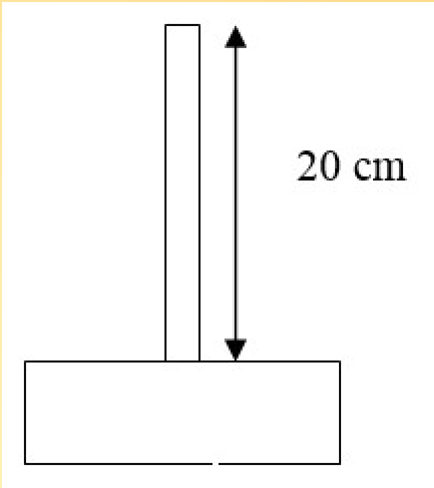
09 Parsial Differential Equation



Difusi alkohol

Suatu tabung panjang 20 cm mula-mula berisi udara dengan 2 % uap alkohol. Pada bagian bawah tabung berhubungan dengan bejana berisi alkohol sehingga alkohol tersebut menguap melalui tabung yang mula-mula berisi udara diam tersebut. Pada bagian ini konsentrasi alkohol dijaga tetap 10 %. Pada bagian atas (puncak) tabung uap alkohol di permukaan atas tabung dapat dianggap selalu nol.

Tentukan distribusi konsentrasi alkohol pada tabung sampai minimal 1000 detik. Diketahui $\vartheta = 0{,}119\ cm^2$ / detik.



Persamaan Parabolik

d^2 c   d c
D -------- = -----
d x^2   d t

Kondisi awal

c(x,0) = 2

Kondisi batas

c(0,t) = 0   c(20,t) = 10

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^{j} - 2u_{i}^{j} + u_{i-1}^{j}}{(\Delta x)^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_{i}^{j+1} - u_{i}^{j}}{\Delta t}$$

$r = \vartheta\, \Delta t/(\Delta x)^2 = 1/2$ dan $\Delta x = 4$ cm.

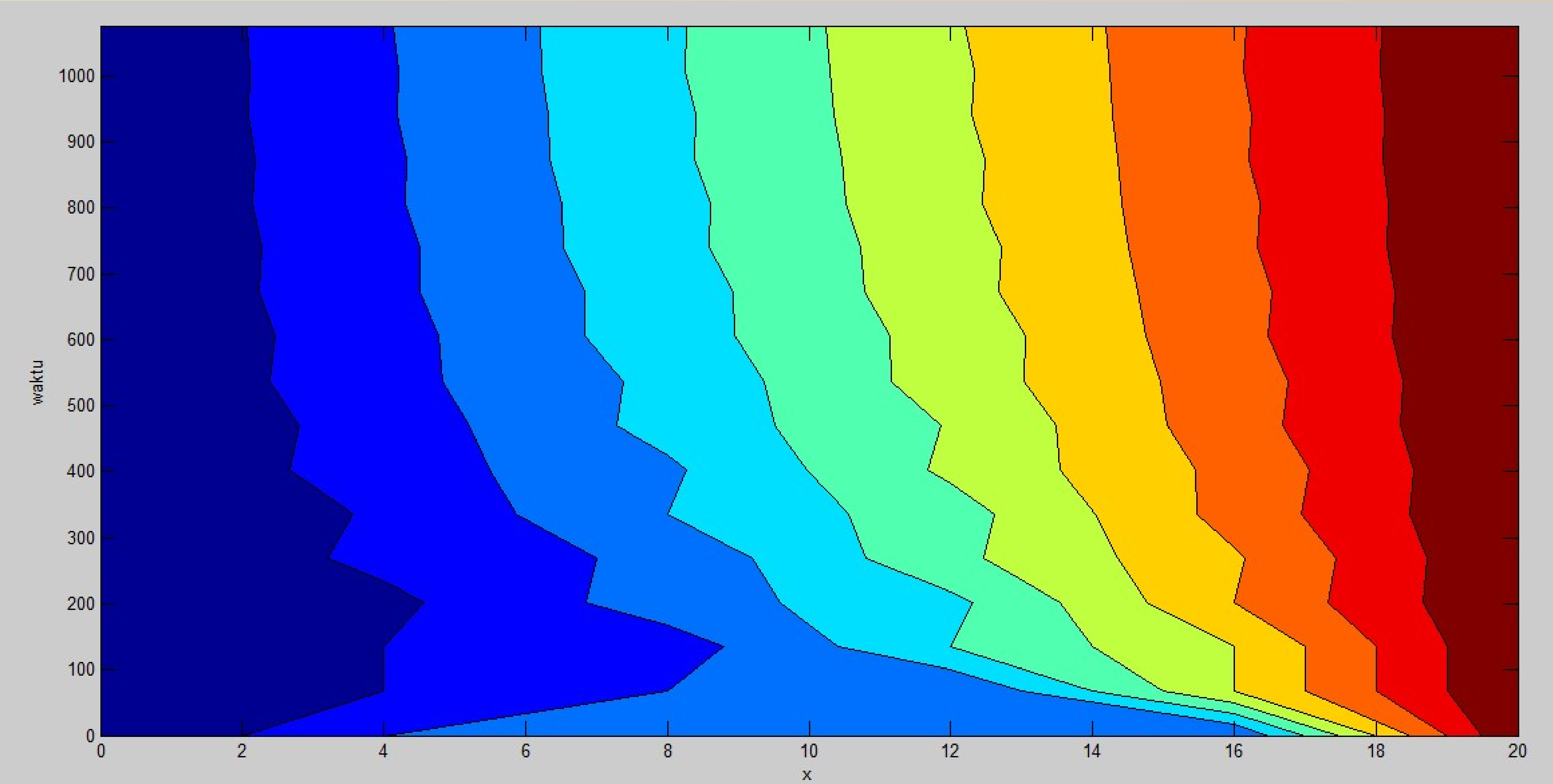
$\Delta t = 0.5(\Delta x)^2/\vartheta = 67{,}2$ detik

```
format short
clc
clear all
% Data-data
L=20;
D=0.119;
N=5;
M=0.5;
delx=L/N;
delt=M/D*delx^2;
xo=0;
Jend=16;
for i=1:N+1
            x(i)=xo+delx*(i-1);
end
x
%Kondisi awal
u(1,1)=0.0;
for i=2:N
   u(1,i)=2;
end
u(1,N+1)=10.0;
```

```
% interval waktu
for i=1:Jend+1
   t(i) = delt*i-delt;
end
t=t'
for j=1:Jend
   u(j+1,1)=0.0;
   for i=2:N

u(j+1,i)=(delt*D/delx^2)*(u
(j,i-1)+u(j,i+1))+...
(1-2*delt*D/delx^2)*u(j,i);
   end
   u(j+1,N+1)=10.0;
end
u
mesh(x, t, u(1:Jend+1,:))
xlabel('x');  ylabel('waktu');
zlabel('konsentrasi')
```

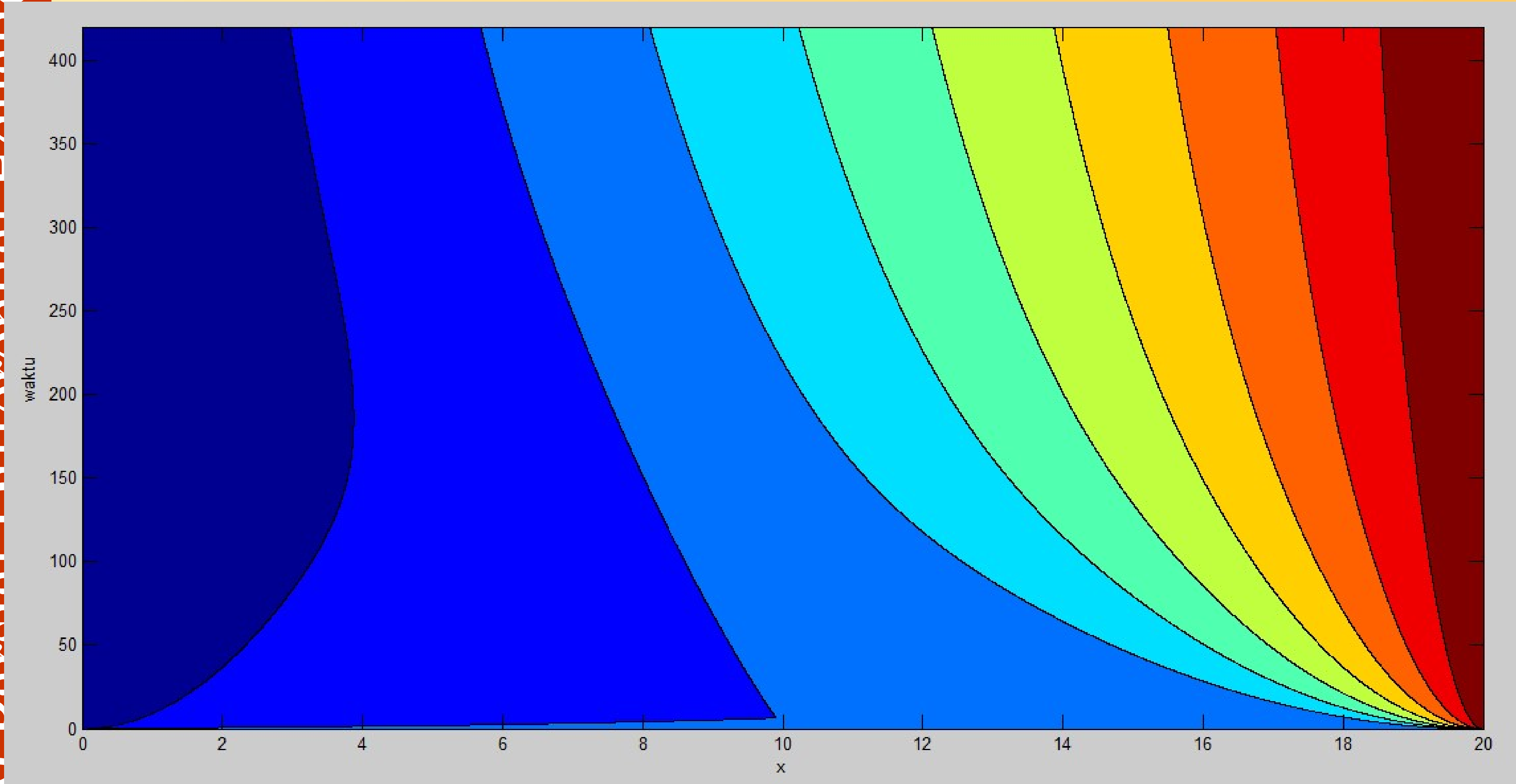
```
x =
     0     4     8    12    16    20

t =
   1.0e+003 *
         0
    0.0672
    0.1345
    0.2017
    0.2689
    0.3361
    0.4034
    0.4706
    0.5378
    0.6050
    0.6723
    0.7395
    0.8067
    0.8739
    0.9412
    1.0084
    1.0756

u =
         0    2.0000    2.0000    2.0000    2.0000   10.0000
         0    1.0000    2.0000    2.0000    6.0000   10.0000
         0    1.0000    1.5000    4.0000    6.0000   10.0000
         0    0.7500    2.5000    3.7500    7.0000   10.0000
         0    1.2500    2.2500    4.7500    6.8750   10.0000
         0    1.1250    3.0000    4.5625    7.3750   10.0000
         0    1.5000    2.8438    5.1875    7.2813   10.0000
         0    1.4219    3.3438    5.0625    7.5938   10.0000
         0    1.6719    3.2422    5.4688    7.5313   10.0000
         0    1.6211    3.5703    5.3867    7.7344   10.0000
         0    1.7852    3.5039    5.6523    7.6934   10.0000
         0    1.7520    3.7188    5.5986    7.8262   10.0000
         0    1.8594    3.6753    5.7725    7.7993   10.0000
         0    1.8376    3.8159    5.7373    7.8862   10.0000
         0    1.9080    3.7875    5.8511    7.8687   10.0000
         0    1.8937    3.8795    5.8281    7.9255   10.0000
         0    1.9398    3.8609    5.9025    7.9140   10.0000
```

09 Parsial Differential Equation

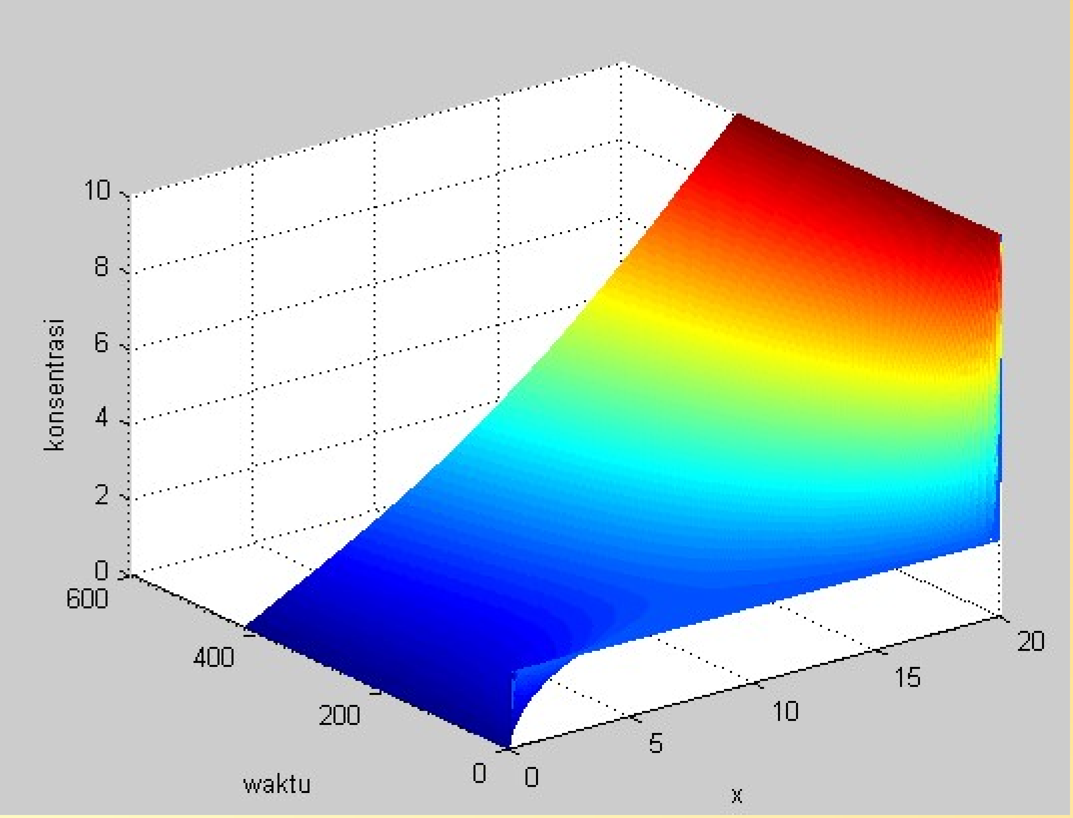


- Coba untuk $\Delta x = 0.1$ cm

09 Parsial Differential Equation

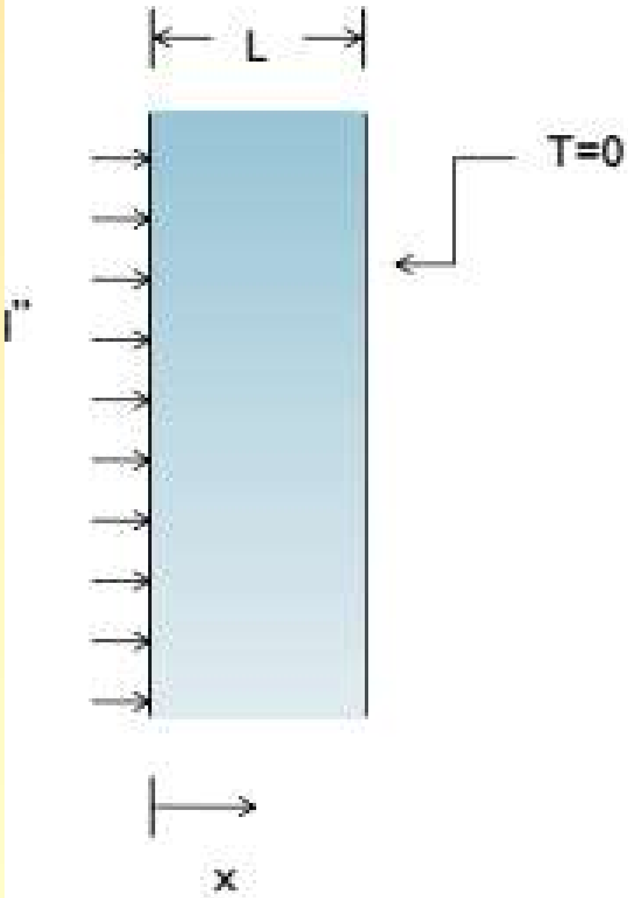


09 Parsial Differential Equation



PDE tool

Model Problem



$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = 0$$

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q''$$

$$T(L,t) = 0$$

pdepe Solves the Following

m=0 for Cartesian, 1 for cylindrical, 2 for spherical

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^{m} f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$u(x,t_0) = u_0(x)$$

Initial Conditions

$$p(x,t,u) + q(x,t) f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0$$

Boundary Conditions – one at each boundary

pdepe Solves the Following

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$c = \rho c_p$$

$$f = k\frac{\partial T}{\partial x}$$

$$s = 0$$

Differential Equations

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
```

Initial Conditions

```
function u0 = pdex1ic(x)
u0 = 0;
```

pdepe Solves the Following

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q'' \qquad p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

or

$$q''+k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad \leftarrow \quad x=0,\; p=q'',\; q=1$$

remember

$$f = k\frac{\partial T}{\partial x}$$

$$T(L,t)=0 \quad \leftarrow \quad x=L,\; p=T=ur,\; q=0$$

Boundary Conditions

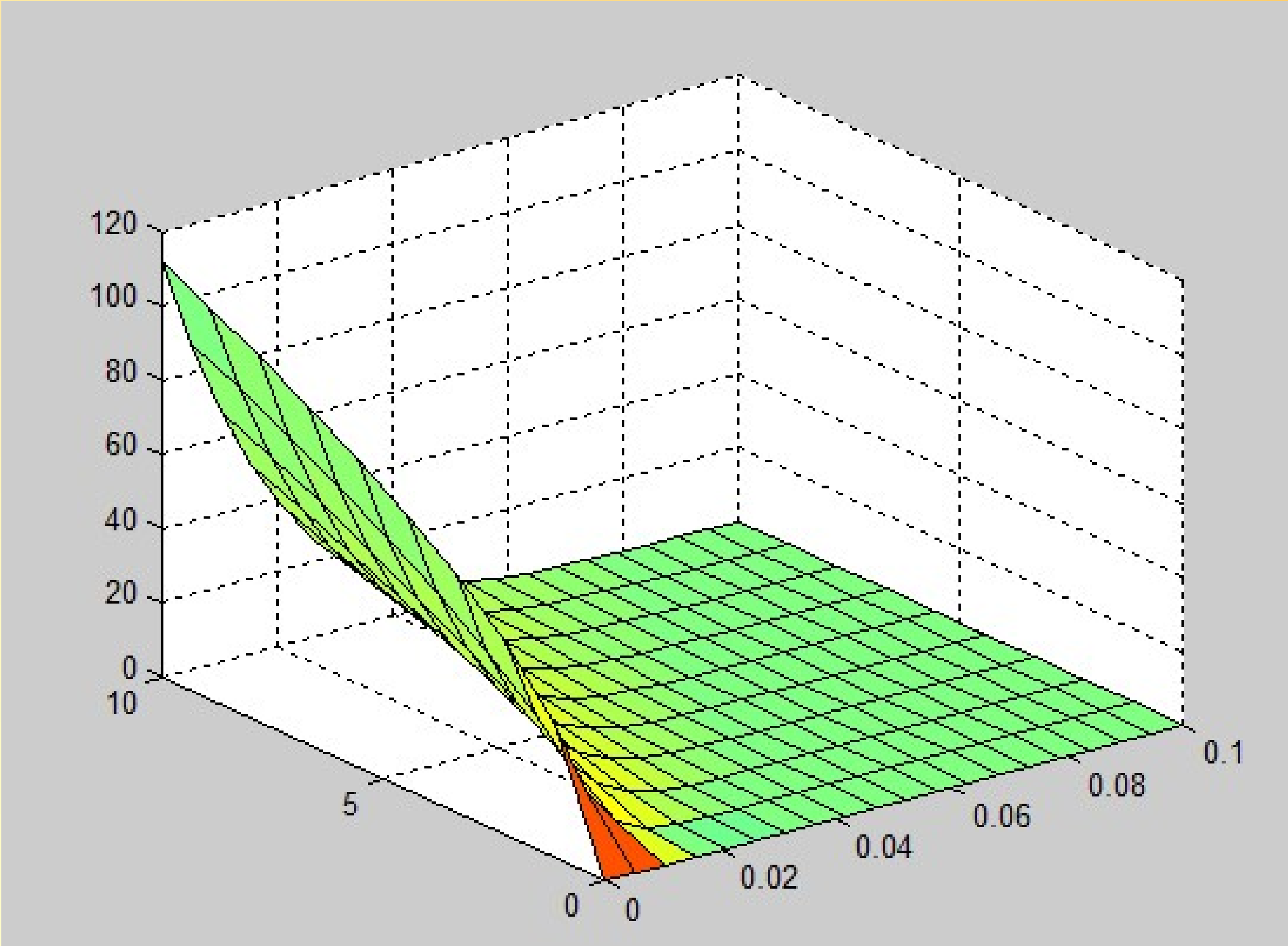
```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q;
ql = 1;
pr = ur;
qr = 0;
```

At left edge, q+k*dT/dx=0

At right edge, ur=0

```
function parabolic
global rho cp k
global q
L=0.1 %m
k=200 %W/m-K
rho=10000 %kg/m^3
cp=500 %J/kg-K
q=1e6 %W/m^2
tend=10 %seconds
m = 0;
x = linspace(0,L,20);
t = linspace(0,tend,10);
sol = pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
surf(x,t,sol)
```

```
-----------------------------------------------------------------
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
-----------------------------------------------------------------
function u0 = pdex1ic(x)
u0 = 0;
-----------------------------------------------------------------
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q; %these two set k*dT/dx-q=0 on right side
ql = 1;
pr = ur;
qr = 0; %sets right side temperature to 0
```


120
100
80
60
40
20
0
10
5
0
0
0.02
0.04
0.06
0.08
0.1