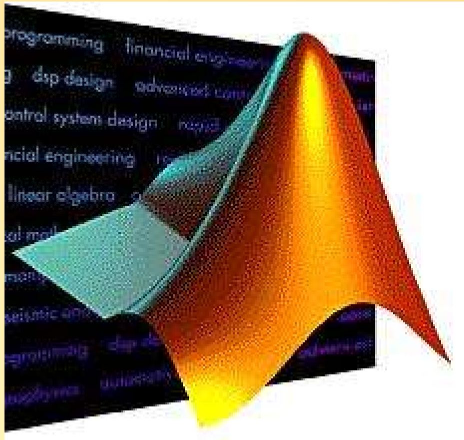
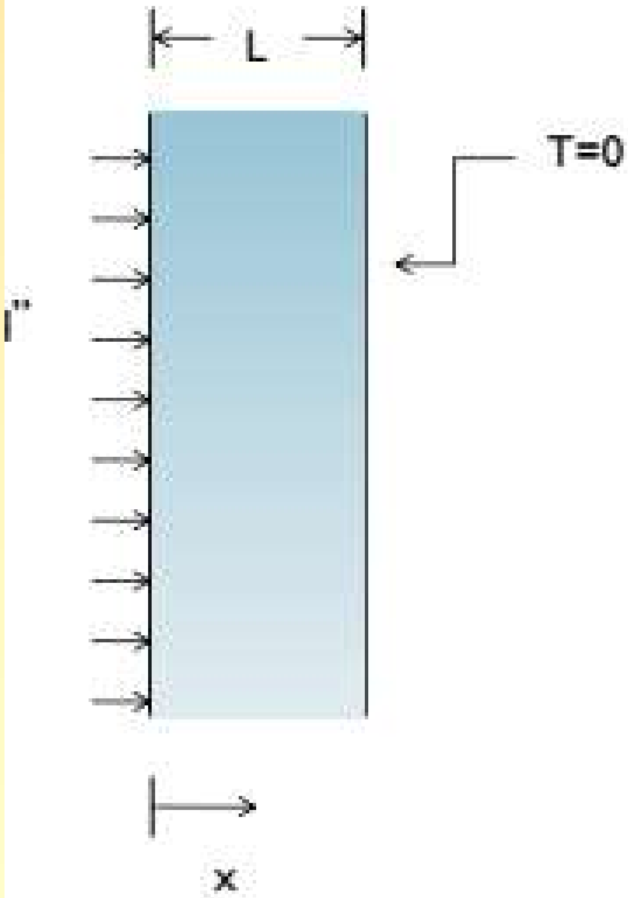
Parsial Differential Equation



PDE tool

Model Problem



$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = 0$$

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q''$$

$$T(L,t) = 0$$

pdepe Solves the Following

m=0 for Cartesian, 1 for cylindrical, 2 for spherical

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$u(x,t_0) = u_0(x)$$

Initial Conditions

$$p(x,t,u) + q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0$$

Boundary Conditions – one at each boundary

pdepe Solves the Following

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$c = \rho c_p$$

$$f = k\frac{\partial T}{\partial x}$$

$$s = 0$$

Differential Equations

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
```

Initial Conditions

```
function u0 = pdex1ic(x)
u0 = 0;
```

pdepe Solves the Following

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q'' \qquad p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

or

$$q''+k\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

$x = 0$

$p = q''$

$q = 1$

remember

$$f = k\frac{\partial T}{\partial x}$$

$x = L$

$p = T = ur$

$q = 0$

$$T(L,t) = 0$$

Boundary Conditions

```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q;
ql = 1;
pr = ur;
qr = 0;
```

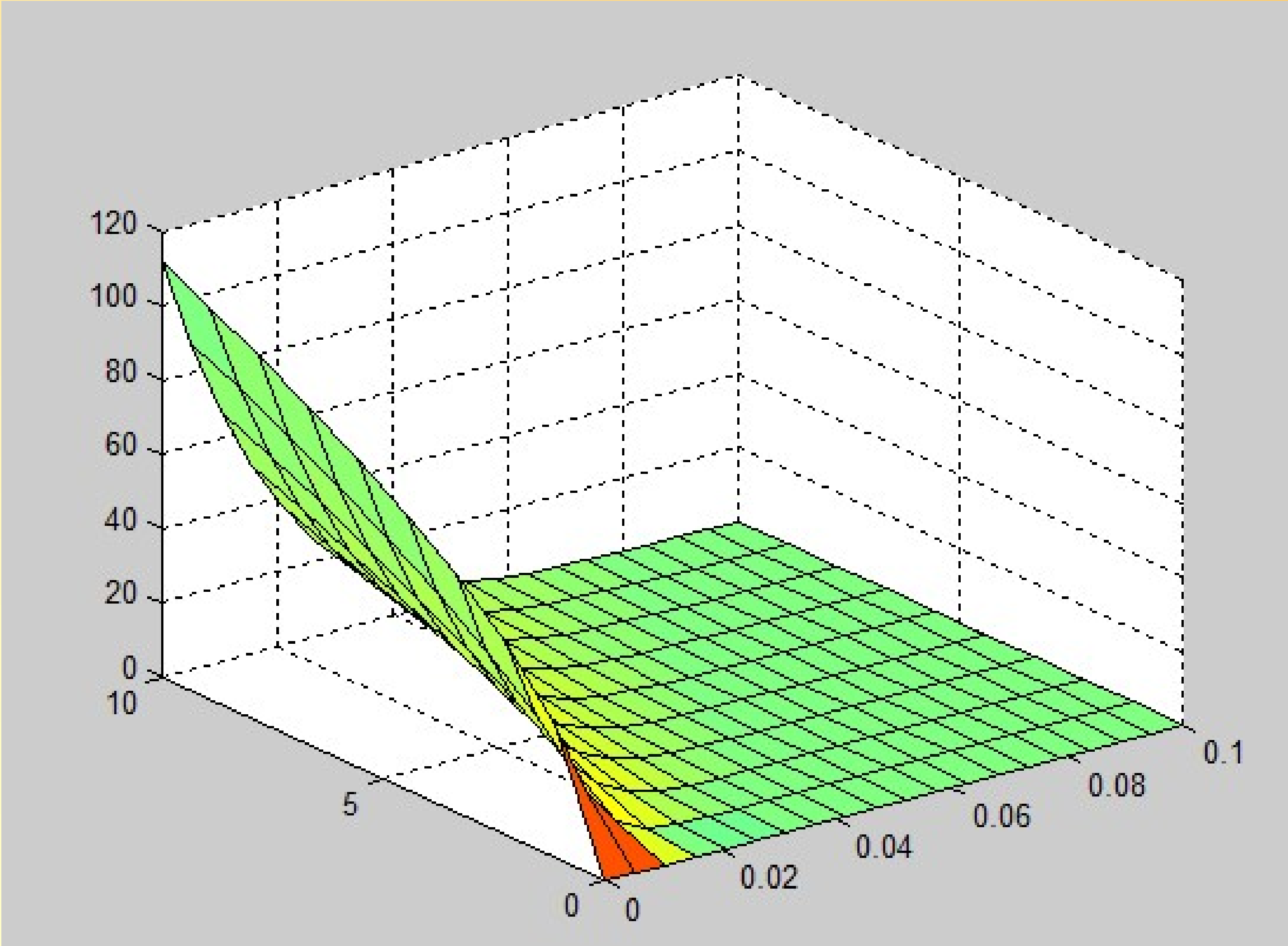
At left edge, q+k*dT/dx=0

At right edge, ur=0

```
function parabolic
global rho cp k
global q
L=0.1 %m
k=200 %W/m-K
rho=10000 %kg/m^3
cp=500 %J/kg-K
q=1e6 %W/m^2
tend=10 %seconds
m = 0;
x = linspace(0,L,20);
t = linspace(0,tend,10);
sol = pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
surf(x,t,sol)
```

```
-----------------------------------------------------------------
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
-----------------------------------------------------------------
function u0 = pdex1ic(x)
u0 = 0;
-----------------------------------------------------------------
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q; %these two set k*dT/dx-q=0 on right side
ql = 1;
pr = ur;
qr = 0; %sets right side temperature to 0
```

09 Parsial Differential Equation



PDEtool

equation solver such as **ode45**. The following specific PDE can be solved with **pdepe**:

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left[x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right] + s\left(x,t,u,\frac{\partial u}{\partial x}\right) \quad (11.3.1)$$

In addition to the PDE, boundary conditions must also be specific. The specific form required for **pdepe** can be nonlinear and time dependent so that:

$$p(x,t,u) + q(x,t)g(x,t,u,u_x) = 0 \text{ at } x = a, b\,. \quad (11.3.2)$$

PDEtool

To start, consider the simplest PDE: the heat equation:

$$\pi^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{11.3.3}$$

where the solution is defined on the domain $x \in [0, 1]$ with the boundary conditions

$$u(0, t) = 0 \tag{11.3.4a}$$

$$\pi \exp(-t) + \frac{\partial u(1, t)}{\partial t} = 0 \tag{11.3.4b}$$

and initial conditions

$$u(x, 0) = \sin(\pi x)\,. \tag{11.3.5}$$

M-file : Main command

```
m = 0;
x = linspace(0,1,20);
t = linspace(0,2,5);
u = pdepe(m,'pdex1pde','pdex1ic','pdex1bc',x,t);
surf(x,t,u)
```

Function pdex1pde

$$\pi^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

equation solver such as **ode45**. The following specific PDE can be solved with **pdepe**:

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left[x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right] + s\left(x,t,u,\frac{\partial u}{\partial x}\right) \quad (11.3.1)$$

In this case, the implementation is fairly straightfoward since $s(x,t,u,u_x) = 0$, $m = 0$, $c(x,t,u,u_x) = \pi^2$ and $f(x,t,u,u_x) = u_x$. The initial condition is quite easy and can be done in one line.

pdex1pde.m

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
c = pi^2;
f = DuDx;
s = 0;
```

The initial condition is quite easy and can be done in one line

and initial conditions

$$u(x, 0) = \sin(\pi x) \, .$$

pdex1ic.m

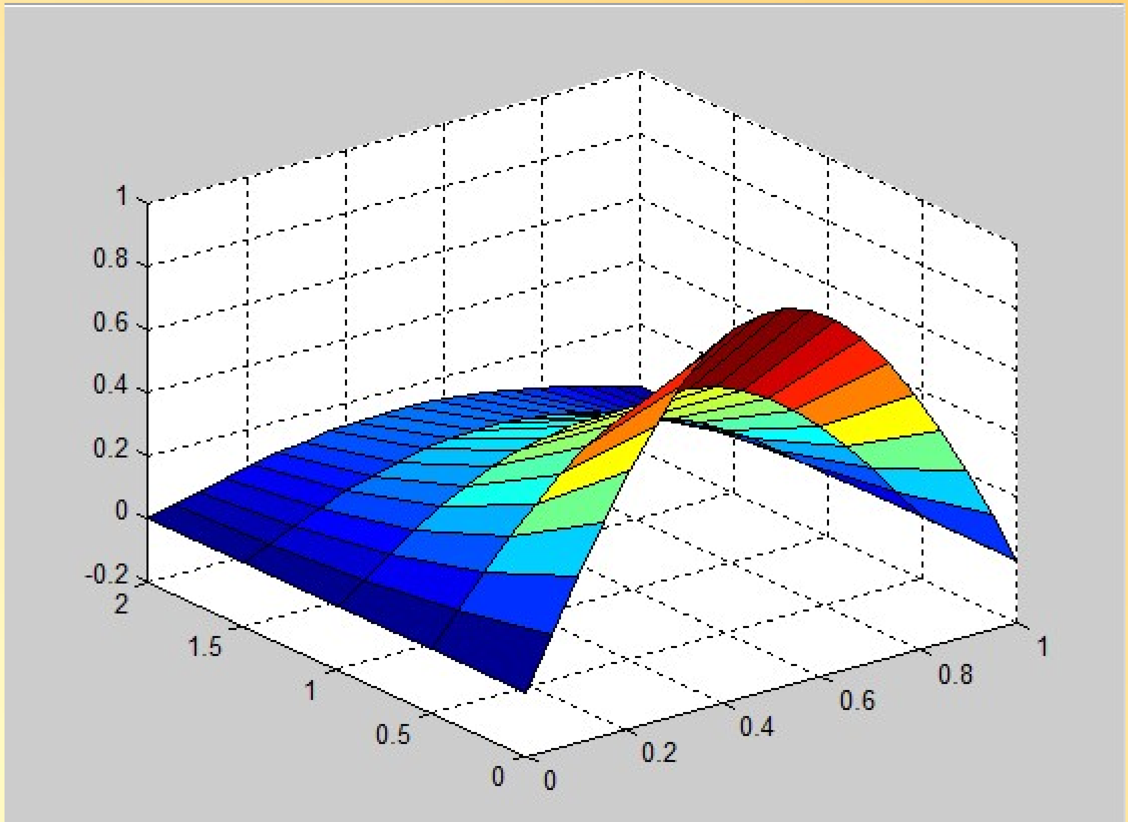
```
function u0 = pdex1ic(x)
u0 = sin(pi*x);
```

the left and right boundaries can be implemented by specifying the function q, q and g at the right and left

$$u(0,t) = 0$$

$$\pi \exp(-t) + \frac{\partial u(1,t)}{\partial t} = 0$$

```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr = pi * exp(-t);
qr = 1;
```


1
0.8
0.6
0.4
0.2
0
-0.2
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

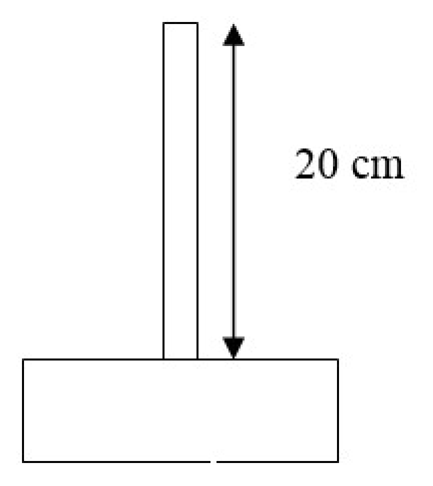
- Selesaikan soal sebelumnya dg PDEtools

Difusi alkohol

Suatu tabung panjang 20 cm mula-mula berisi udara dengan 2 % uap alkohol. Pada bagian bawah tabung berhubungan dengan bejana berisi alkohol sehingga alkohol tersebut menguap melalui tabung yang mula-mula berisi udara diam tersebut. Pada bagian ini konsentrasi alkohol dijaga tetap 10 %. Pada bagian atas (puncak) tabung uap alkohol di permukaan atas tabung dapat dianggap selalu nol.

Tentukan distribusi konsentrasi alkohol pada tabung sampai minimal 1000 detik.

Diketahui $\vartheta = 0{,}119\ cm^2$ / detik.



Persamaan Parabolik

$$D \frac{d^2 c}{d x^2} = \frac{d c}{d t}$$

Kondisi awal

$c(x,0) = 2$

Kondisi batas

$c(0,t) = 0$ $\quad$ $c(20,t) = 10$

pdepe Solves the Following

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

m=0 for Cartesian, 1 for cylindrical, 2 for spherical

$$u(x,t_0) = u_0(x)$$

Initial Conditions

$$p(x,t,u) + q(x,t) f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0$$

Boundary Conditions – one at each boundary

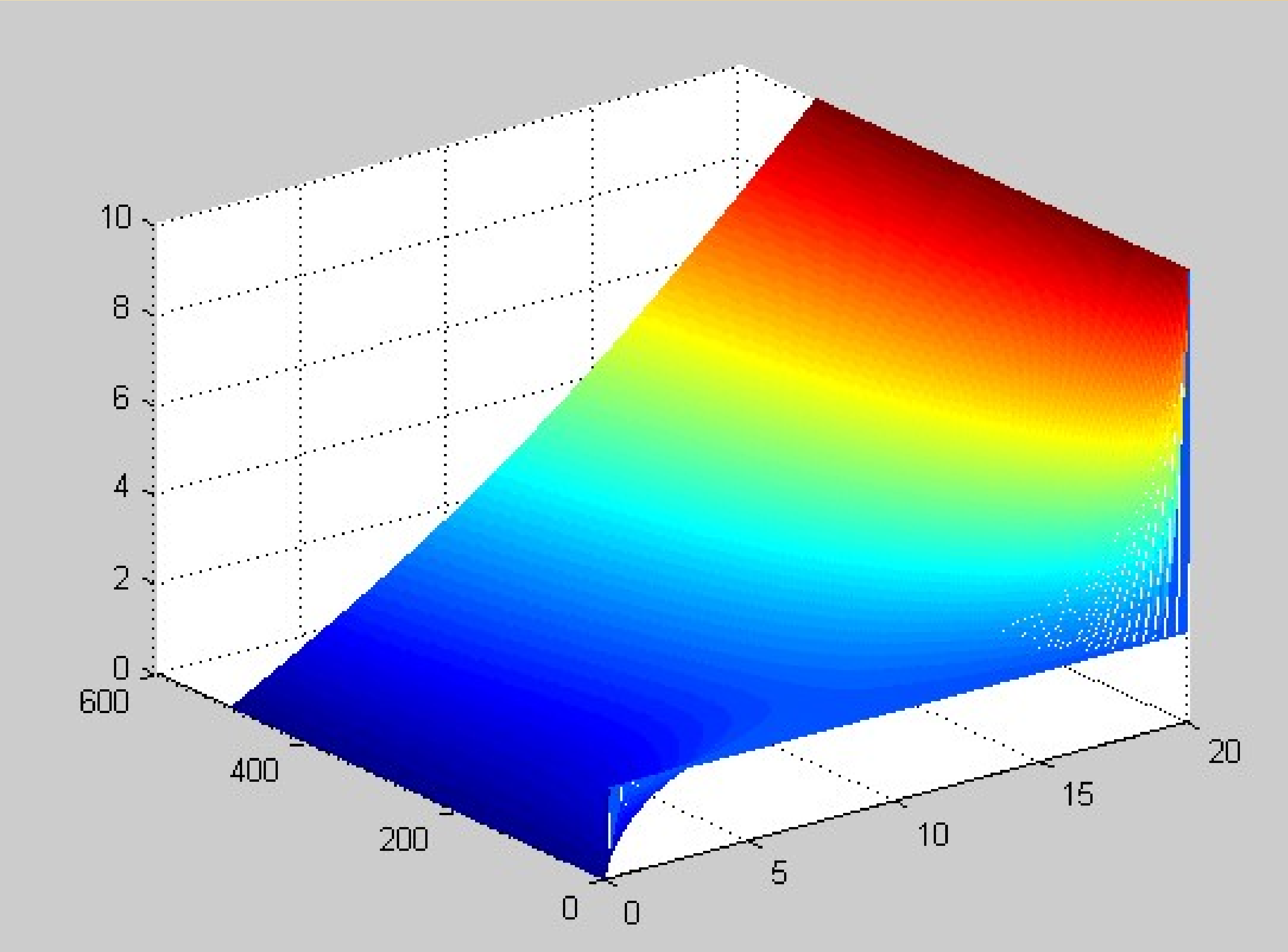
The Main file

```
m = 0;
x = linspace(0, 20, 201);
t = linspace(0, 500, 501)
u = pdepe(m,'pdex3pde','pdex3ic','pdex3bc',x,t);
mesh(x,t,u)
```

```
function [c,f,s] = pdex3pde(x,t,u,DuDx)
c = 1/0.119;
f = DuDx;
s = 0;
```

```
function u0 = pdex3ic(x)
u0 = 2;
```

```
function [pl,ql,pr,qr] = pdex3bc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr = ur-10;
qr = 0;
```


10
8
6
4
2
0
600
400
200
0
0
5
10
15
20

the transient behavior of a rod at constant T put between two heat reservoirs at different temperatures, again T1 = 100, and T2 = 200. The rod will start at 150. Over time, we should expect a solution that approaches the steady state solution: a linear temperature profile from one side of the rod to the other.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

t = 0; To(x) = 150

T(0, t) = 100

T(L,t) = 200

```
m=0; % specifies 1-D symmetry
x = linspace(0,1); % equally-spaced points along the length of the rod
t = linspace(0,5); % this creates time points
sol = pdepe(m,@pdex,@pdexic,@pdexbc,x,t);
surf(x,t,sol)
xlabel('Position')
ylabel('time')
zlabel('Temperature')
```

```
function [c,f,s] = pdex(x,t,u,DuDx)
c = 1;
f = 0.02*DuDx;
s = 0;



function u0 = pdexic(x)
u0 = 150;



function [pl,ql,pr,qr] = pdexbc(xl,ul,xr,ur,t)
pl = u1-100;
ql = 0;
pr= ur-200;
qr = 0;
```

A Second Problem

- Suppose we want convection at x=L
- That is

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

$$-k\frac{dT}{dx}=h(T-T_{bulk})$$

or

$$hT-hT_{bulk}+k\frac{dT}{dx}=0$$

$$x=L$$

$$p=T=h(ur-T_{bulk})$$

$$q=1$$

Diffusion Equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

Initial Condition

$$c(x,0) = 0$$

Boundary Conditions

$$c(0,t) = C_{max}$$

$$\frac{\partial c}{\partial x}(L,t) = 0$$

09 Parsial Differential Equation

The diffusion equation as derived earlier can be written as

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} \qquad (37.4)$$

Initial condition

(37.5)

$$C(x, t = 0) = 0$$

Boundary conditions

(37.6)

$$C(x = 0, t) = C_o$$

$$C(x = \infty, t) = 0$$

(37.7)

Comparing (37.4) with (37.1), we have

$$g\left(x, t, c, \frac{\partial c}{\partial x}\right) = 1$$

(37.6)

$$f\left(x, t, c, \frac{\partial c}{\partial x}\right) = D_x \frac{\partial c}{\partial x} \qquad (37.7)$$

$$s\left(x, t, c, \frac{\partial c}{\partial x}\right) = 0$$

(37.8)

$$m = 1$$

(37.9)

Comparing (37.5) with (37.2), we have

$$\mathrm{co} = 0 \tag{37.10}$$

Comparing (37.6) and (37.7) with (37.2), we have

$$pl = cl - 1 \tag{37.11}$$

$$ql = 0 \tag{37.12}$$

$$pr = cr \tag{37.13}$$

$$qr = 0 \tag{37.14}$$

```
function [g,f,s] = pdefun_ex1(x,t,c,DcDx)
      D=0.0000972;
      g = 1;
      f = D*DcDx;
      s = 0;
end
```

Function icfun

```
function c0 = icfun_ex1(x)
      c0 = 0;
end
```

Function bcfun

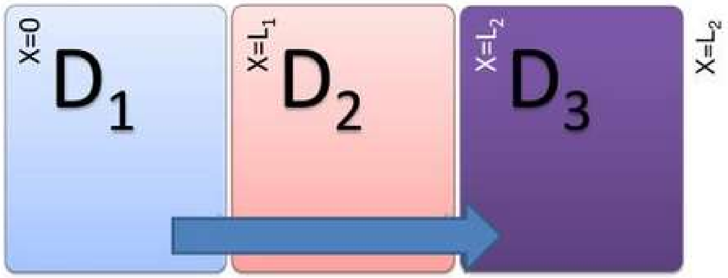
```
function [pl,ql,pr,qr] = bcfun_ex1(xl,cl,xr,cr,t)
      pl = cl-1;
      ql = 0;
      pr = cr;
      qr = 0;
end
```

```
clear all;
Close all;

m = 0;
x = linspace(0,2.5,200); % Spatial discretization
t = linspace(0,730,100); % Temporal descretization

sol = pdepe(m,@pdefun_ex1,@icfun_ex1,@bcfun_ex1,x,t);

% The following loop is for plotting the result

for i=10:10:100
    plot(x,sol(i,:),'color', [rand rand rand]);
    axis tight;
    xlabel('X');
    ylabel('C/Co');
    hold on;
end
```

```
%----------------------functions-------------------------------
function [c,f,s] = transfun(x,t,u,DuDx,D,v,c0,cin)
c = 1;
f = D*DuDx;
s = -v*DuDx;
% ------------------------------------------------------------
function u0 = ictransfun(x,D,v,c0,cin)
u0 = c0;
% ------------------------------------------------------------
function [pl,ql,pr,qr] = bctransfun(xl,ul,xr,ur,t,D,v,c0,cin)
pl = ul-cin;
ql = 0;
pr = 0;
qr = 1;
```

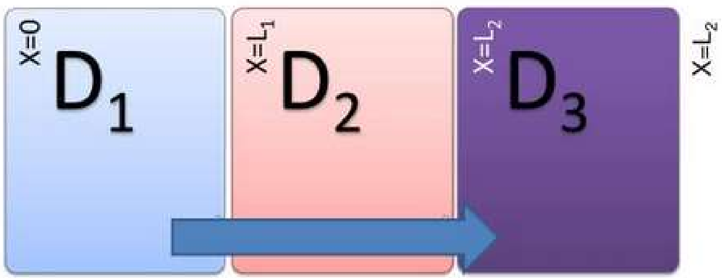
```
function pdepetrans
% transport-solver using 'pdepe'

T = 4.;                        % maximum time [s]
L = 2.5;                       % length [m]
D = 0.01;                      % diffusivity [m*m/s]
v = 1;                         % velocity [m/s]
c0 = 0;                        % initial concentration [kg/m*m*m]
cin = 1;                       % boundary concentration [kg/m*m*m]
M = 100;                       % number of timesteps
N = 100;                       % number of nodes

t = linspace (T/M,T,M);        % time discretization
x = linspace (0,L,N);          % space discretization

%----------------------execution---------------
options = odeset;
c = pdepe (0,@transfun,@ictransfun,@bctransfun,x,…
 [0 t],options,D,v,c0,cin);

%---------------------- output ----------------
plot ([0 t],c)                  % breakthrough curves
      xlabel ('time'); ylabel ('concentration');
```

Example 1: A Mass Transfer System



$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

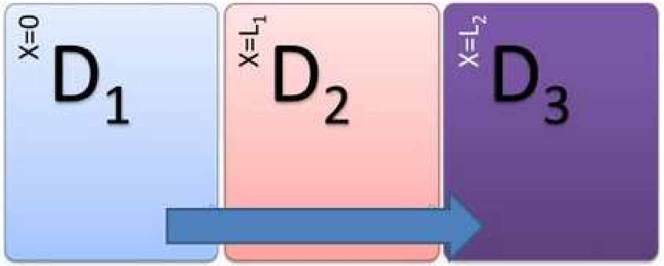
Initial Conditions



@ t=0

$0 \le x \le L_1$	$L_1 \le x \le L_2$	$L_2 \le x \le L_3$
$C = C_0$	$C = 0$	$C = 0$

Boundary Conditions



$$x = 0 \qquad \frac{\partial C}{\partial x} = 0$$

$$x = L_3 \qquad \frac{\partial C}{\partial x} = 0$$

$$x = L_1 \qquad -D_1 \frac{\partial C}{\partial x} = -D_2 \frac{\partial C}{\partial x}$$

$$x = L_2 \qquad -D_2 \frac{\partial C}{\partial x} = -D_3 \frac{\partial C}{\partial x}$$

7

$$c(x,t,u,\frac{\partial u}{\partial x})\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}(x^m b(x,t,u,\frac{\partial u}{\partial x})) + s(x,t,u,\frac{\partial u}{\partial x})$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

$$c(x,t,u,u_x) = 1$$

$$m = 0$$

$$b(x,t,u,u_x) = D_1.\frac{\partial C}{\partial x} = D_1.u_x$$

$$s(x,t,u,u_x) = 0$$

$$x = 0, \quad \frac{\partial C}{\partial x} = 0$$

$$b = D_1 . \frac{\partial C}{\partial x}$$

$$p(xl, t, u) + q(xl, t).b(xl, t, u, u_x) = 0$$

$$\boxed{p(0, t, u) = 0} \qquad \boxed{q(0, t) = \frac{1}{D_1}}$$

$$x = L_1, \quad -D_1 \frac{\partial C}{\partial x} = -D_2 \frac{\partial C}{\partial x} \text{ or}$$

$$x = L_1, \quad \frac{(D_2 - D_1)}{D_1} D_1 \frac{\partial C}{\partial x} = 0$$

$$\boxed{D_1 . \frac{\partial C}{\partial x}}$$

$$p(xr, t, u) + q(xr, t).b(xr, t, u, u_x) = 0$$

$$\boxed{p(L_1, t, u) = 0} \qquad \boxed{q(L_1, t) = \frac{D_2 - D_1}{D_1}}$$

$$u(0, x) = f(x)$$

$$C(0, x) = C_0$$

$$u(0, x) = C_0$$

09 Parsial Differential Equation

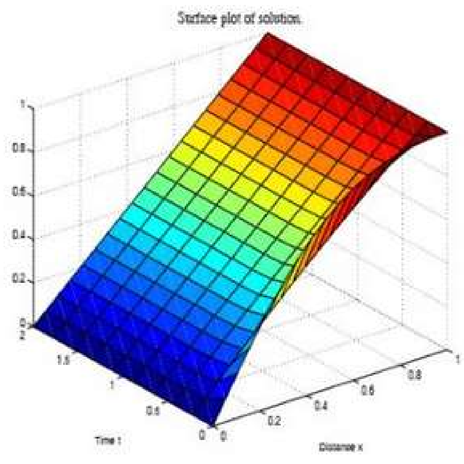
```
function [c,b,s] = system(x,t,u,DuDx)
    c = 1;
    b =D1*DuDx;
    s = 0;
end
```

```
function [pl,ql,pr,qr] =
 bc1(xl,ul,xr,ur,t)

pl = 0;
ql = 1/D1;
pr = 0;
qr = (D2-D1)/D1;
end
```

```
function value = initial1(x)

   value = C0;

end
```

```
m = 0;
%Define the solution mesh
x = linspace(0,1,20);
t = linspace(0,2,10);
%Solve the PDE
u = pdepe(m,@system,@initial1,@bc1,x,t);
%Plot solution
surf(x,t,u);
title('Surface plot of solution.');
xlabel('Distance x');
ylabel('Time t');
```

Results



Example 2.39: One-Dimensional Parabolic PDE

The temperature $u(x,t)$ in a wall of unit length can be described by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

The thickness of the wall is 1 m and the initial profile of the temperature in the wall at $t = 0$ sec is uniform at $T = 90$°C. At time $t = 0$, the ambient temperature is suddenly changed to 15°C and held there. If we assume that there is no convection resistance, the temperature of both sides of the wall is also held constant at 15°C. Determine the temperature distribution graphically within the wall from $t = 0$ to $t = 21{,}600$ sec. The wall property can be assumed as $\alpha = 4.8 \times 10^{-7}$ m/sec^2.

Solution

Step 1: From the heat equation, we have $((1/\alpha)(\partial u/\partial t)) = (\partial^2 u/\partial x^2)$. Thus, we can see that $m = 0$, $g = 1/\alpha$, $f = \partial u/\partial x$, and $r = 0$. Create a function that holds g, f, and r:

```
function [g,f,r] = pdeTde(x,t,u,DuDx)
alpha = 4.8e-7;
g = 1/alpha;
f = DuDx;
r = 0;
end
```

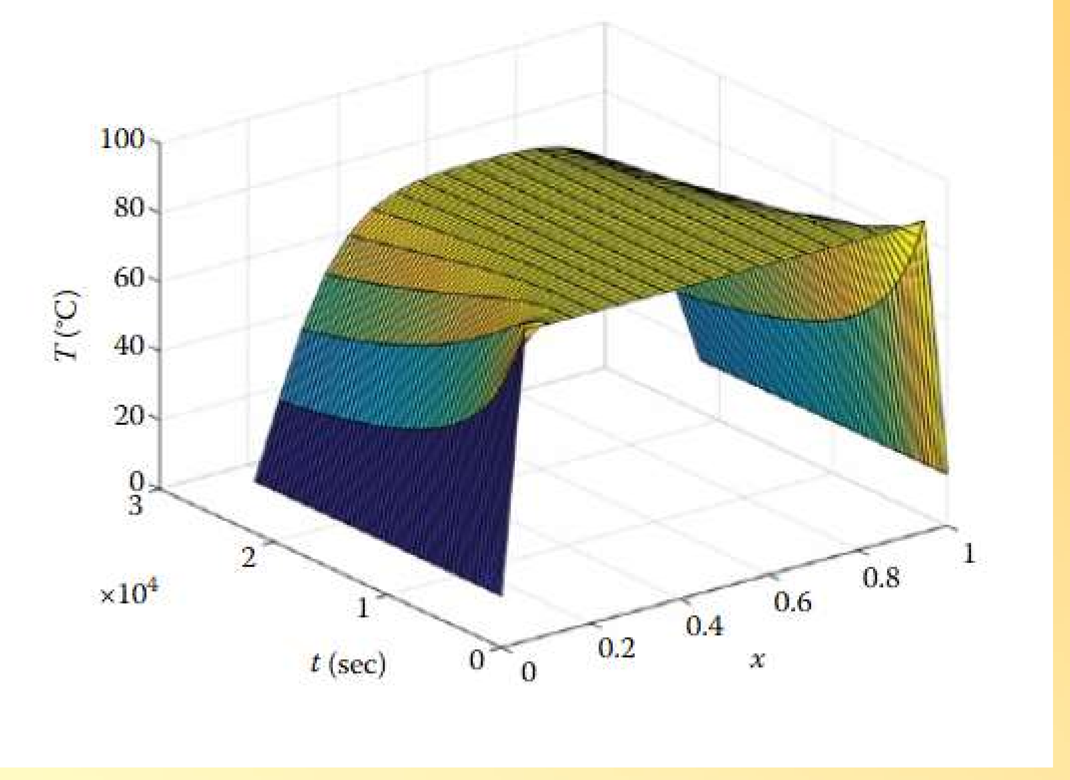
```
function u0 = pdeTic(x)
u0 = 90;
end
```

Step 3: Specify the boundary conditions $p(x, t, u) + q(x, t)f(x, t, u, (\partial u/\partial x)) = 0$. Since the temperature at both sides of the wall is 15°C, $p = pl = ul - 15$ and $q = ql = 0$ at $x = 0$, and $p = pr = ur - 15$ and $q = qr = 0$ at $x = 1$. Create a function that holds these boundary conditions.

```
function [pl,ql,pr,qr] = pdeTbc(xl,ul,xr,ur,t)
pl = ul-15;
ql = 0;
pr = ur-15;
qr = 0;
end
```

Step 4: Set the intervals for x and t, and call the function *pdepe* to solve the equation. Suppose that the range of x is divided into 20 subintervals and the time span is divided into 180 subintervals. The following script generates the temperature profile in the wall as shown in Figure 2.21:

```
>> m = 0; x = linspace(0,1,20);
>> t = linspace(0,21600,54);
```


100
80
60
40
20
0
T (°C)
3
2
1
0
×10^4
t (sec)
0
0.2
0.4
0.6
0.8
1
x

next

- MATLAB can perform symbolic calculus on expressions. Consider the following example:

```
syms x
f=sin(x^2)
f =
sin(x^2)
diff(f,x)
ans =
2*x*cos(x^2)
```

```
syms x y
q=x^2*y^3*exp(x)
q =
x^2*y^3*exp(x)
pretty(q)
  2  3
x  y  exp(x)
diff(q,y)
ans =
3*x^2*y^2*exp(x)
```

```
syms a t
u=exp(a*t)
u =
exp(a*t)
diff(u,t)
ans =
a*exp(a*t)
```

check whether the function $u(t)=e^{at}$ is a solution of the ODE

$$\frac{du}{dt} - au = 0.$$

syms a t

u=exp(a*t)

u =

exp(a*t)

diff(u,t)-a*u

ans =

0

$w(x,y)=\sin(\pi x)+\sin(\pi y)$ a solution of the differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0?$$

```
syms x y
w=sin(pi*x)+sin(pi*y)
w =
sin(pi*x) + sin(pi*y)
diff(w,x,2)+diff(w,y,2)
ans =
- pi^2*sin(pi*x) - pi^2*sin(pi*y)
simplify(ans)
ans =
-pi^2*(sin(pi*x) + sin(pi*y))
```

Since the result is not zero, the function w is not a solution of the PDE

To compute a mixed partial derivative, we have to iterate the **diff** command. Here is the mixed partial derivative of $w(x,y)=x^2+xy^2$ with respect to x and then y:

```
syms x y
w=x^2*exp(y)+x*y^2
w =
x^2*exp(y) + x*y^2
diff(diff(w,x),y)
ans =
2*y + 2*x*exp(y)
```

can use functions. Consider the following

```
clear
syms a x
f=@(x)exp(a*x)
f =
@(x)exp(a*x)
f(1)
diff(f(x),x)-a*f(x)
ans =
0
```

```
syms x
f=x^2
f =
x^2
int(f,x)
ans =
x^3/3
```

To compute a definite integral

```
int(x^2,x,0,1)
ans =
1/3
```

```
f=@(x)exp(cos(x))
f =
@(x)exp(cos(x))

quad(f,0,1)
ans =
2.3416
```

"quad" is short for quadrature, another term for numerical integration

$$\frac{d}{dx}\int_c^d F(x,y)\,dy = \int_c^d \frac{\partial F}{\partial x}(x,y)\,dy.$$

```
syms x y c d
f=x*y^3+x^2*y
f =
x^2*y + x*y^3
r1=diff(int(f,y,c,d),x)
r1 =
- (x*(c^2 - d^2))/2 - ((c^2 - d^2)*(c^2 + d^2 + 2*x))/4
r2=int(diff(f,x),y,c,d)
r2 =
-((c^2 - d^2)*(c^2 + d^2 + 4*x))/4
r1-r2
ans =
((c^2 - d^2)*(c^2 + d^2 + 4*x))/4 - ((c^2 - d^2)*(c^2 + d^2 + 2*x))/4 -
(x*(c^2 - d^2))/2
simplify(ans)
ans =
0
```

The MATLAB solve command

- to solve the linear equation $ax+b=0$ for x.

syms f x a b

f=a*x+b

f =

b + a*x

solve(f,x)

ans =

-b/a

To solve x^2-3*x+2=0

```
syms f x
f=x^2-3*x+2;
solve(f,x)
ans =
1
2
```

to solve the equations

x+y=1

2x-y=1

syms x y

s=solve(x+y-1,2*x-y-1,x,y)

s.x

ans =

2/3

s.y

ans =

1/3

s=solve(x^2+y^2-1,y-x^2,x,y)

s =

x: [4x1 sym]

y: [4x1 sym]

The first solution is

pretty(s.x(1))

pretty(s.y(1))

The second solution is

pretty(s.x(2))

pretty(s.y(2))

```
clear
syms u1 u2 u3
u1=sym([1;0;2]);
u2=sym([0;1;1]);
u3=sym([1;2;-1]);
A=[u1,u2,u3]

A =
[ 1, 0, 1]
[ 0, 1, 2]
[ 2, 1, -1]

b=sym([8;2;-4]);
x=A\b
x =
18/5
-34/5
22/5
```