11

Functional Equations

Equations for unknown functions are called *functional equations*. We dealt with these already in the chapters on sequences and polynomials. Sequences and polynomials are just special functions.

Here are five examples of functional equations of a single variable:

$$f(x) = f(-x), \quad f(x) = -f(-x), \quad f \circ f(x) = x, \quad f(x) = f\left(\frac{x}{2}\right);$$
$$f(x) = \cos\frac{x}{2} f\left(\frac{x}{2}\right), \quad f(0) = 1, \quad f \text{ continuous.}$$

The first three properties characterize even functions, odd functions, and involutions, respectively. Many functions have the fourth property. On the other hand, the last condition makes the solution unique.

Here are examples of famous functional equations in two variables:

$$f(x+y) = f(x) + f(y), \quad f(x+y) = f(x)f(y), \quad f(xy) = f(x) + f(y),$$

and $f(xy) = f(x)f(y)$. These are *Cauchy's functional equations.*

$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$. This is Jensen's functional equation.

$f(x+y) + f(x-y) = 2f(x)f(y)$. This is d'Alembert's functional equation.

$$g(x+y) = g(x)f(y) + f(x)g(y), \quad f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x-y) = g(x)f(y) - g(y)f(x), \quad f(x-y) = f(x)f(y) + g(x)g(y).$$

The last four functional equations are the addition theorems for the trigonometric functions $f(x) = \cos x$ and $g(x) = \sin x$.

Usually a functional equation has many solutions, and it is quite difficult to find all of them. On the other hand it is often easy to find all solutions with some additional properties, for example, all continuous, monotonic, bounded, or differentiable solutions.

Without additional assumptions, it may be possible to find only certain properties of the functions. We give some examples:

E1. First we consider the equation

$$f(xy) = f(x) + f(y). \tag{1}$$

One solution is easy to guess: $f(x) = 0$ for all x. This is the only solution which is defined for $x = 0$. If $x = 0$ belongs to the domain of f, then we can set $y = 0$ in (1), and we get $f(0) = f(x) + f(0)$, implying $f(x) = 0$ for all x. Let $x = 1$ be in the domain of f. With $x = y = 1$, we get $f(1) = 2f(1)$, or

$$f(1) = 0. \tag{2}$$

If both 1 and -1 belong to the domain, then f is an even function, i.e., $f(-x) = f(x)$ for all x. To prove this, we set $x = y = -1$ in (1), and because of (2), we get

$$f(1) - 2f(-1) - 0 \to f(-1) - 0.$$

Setting $y = -1$ in (1), we get $f(-x) = f(x) + f(1)$, or

$$f(-x) = f(x) \quad \text{for all } x.$$

Assume that f is differentiable for $x > 0$. We keep y fixed and differentiate for x. Then we get $yf'(xy) = f'(x)$. For $x = 1$, one gets $yf'(y) = f'(1)$. Change of notation leads to $f'(x) = f'(1)/x$, or

$$f(x) = \int_1^x \frac{f'(1)}{t}\, dt = f'(1) \ln x.$$

If the function is also defined for $x < 0$, then we have $f(x) = f'(1) \ln |x|$.

E2. A famous classical functional equation is

$$f(x + y) = f(x) + f(y). \tag{1}$$

First, we try to get out of (1) as much information as possible without any additional assumptions. $y = 0$ yields $f(x) = f(x) + f(0)$, that is,

$$f(0) = 0. \tag{2}$$

For $y = -x$, we get $0 = f(x) + f(-x)$, or

$$f(-x) = -f(x). \tag{3}$$

Now we can confine our attention to $x > 0$. For $y = x$, we get $f(2x) = 2f(x)$, and by induction,

$$f(nx) = nf(x) \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

For rational $x = \frac{m}{n}$, that is, $n \cdot x = m \cdot 1$, by (4) we get $f(n \cdot x) = f(m \cdot 1)$, $nf(x) = mf(1)$, and

$$f(x) = \frac{m}{n} f(1). \tag{5}$$

If we set $f(1) = c$, then, from (2), (3), (5), we get $f(x) = cx$ for rational x. That is all we can get without additional assumptions.

(a) Suppose f is continuous. If x is irrational, then we choose a rational sequence x_n with limit x. Because of the continuity of f, we have

$$f(x) = \lim_{x_n \to x} f(x_n) = \lim_{x_n \to x} cx_n = cx.$$

Then we have $f(x) = cx$ for all x.

(b) Let f be monotonically increasing. If x is irrational, then we choose an increasing and a decreasing sequence r_n and R_n of rational numbers, which converge toward x. Then we have

$$cr_n = f(r_n) \leq f(x) \leq f(R_n) = cR_n.$$

For $n \to \infty$, both cr_n and cR_n converge to cx. Thus $f(x) = cx$ for all x.

(c) Let f be bounded on $[a, b]$, that is,

$$|f(x)| < M \quad \text{for all } x \in [a, b].$$

We show that f is also bounded on $[0, b - a]$. If $x \in [0, b - a]$, then $x + a \in [a, b]$. From $f(x) = f(x + a) - f(a)$, we get

$$|f(x)| < 2M.$$

If we set $b - a = d$, then f is bounded on $[0, d]$. Let $c = f(d)/d$ and $g(x) = f(x) - cx$. Then

$$g(x - y) = g(x) + g(y).$$

Furthermore, we have $g(d) = f(d) - cd = 0$ and

$$g(x + d) = g(x) + g(d) = g(x),$$

that is, g is periodic with period d. As the difference of two bounded functions, g is also bounded on $[0, d]$. From the periodicity, it follows that g is bounded on the whole number line. Suppose there is an x_0, so that $g(x_0) \neq 0$. Then $g(nx_0) = ng(x_0)$. By choosing n sufficiently large, we can make $|ng(x_0)|$ as large as we want. This contradicts the boundedness of g. Hence, $g(x) = 0$ for all x, that is,

$$f(x) = cx \quad \text{for all } x.$$

In 1905 G. Hamel discovered "wild" functions that are nowhere bounded and also satisfy the functional equation $f(x + y) = f(x) + f(y)$. We are looking for "tame" solutions. If we succeed in finding a solution for all rationals, then we can extend them to reals by continuity or monotonicity, etc.

E3. Another classical equation is

$$f(x+y)=f(x)f(y). \tag{1}$$

If there is an a such that $f(a)=0$, then $f(x+a)=f(x)f(a)=0$ for all x, that is, f is identically zero. For all other solutions, $f(x)\neq 0$ everywhere. For $x=y=t/2$, we get

$$f(t)=f^2\left(\frac{t}{2}\right)>0.$$

The solutions we are looking for are everywhere positive. For $y=0$, we get $f(x)=f(x)f(0)$ from (1), that is, $f(0)=1$. For $x=y$, we get $f(2x)=f^2(x)$, and by induction

$$f(nx)=f^n(x). \tag{2}$$

Let $x=\frac{m}{n}$ $(m,n\in\mathbb{N})$, that is, $n\cdot x=m\cdot 1$. Applying (2), we get $f(nx)=f(m\cdot 1)\Rightarrow f^n(x)=f^m(1)\Rightarrow f(x)=f^{\frac{m}{n}}(1)$. If we set $f(1)=a$, then

$$f\left(\frac{m}{n}\right)=a^{\frac{m}{n}},$$

that is, $f(x)=a^x$ for rational x. With a weak additional assumption (continuity, monotonicity, boundedness), as in **E2**, we can show that

$$f(x)=a^x \quad \text{for all } x.$$

The following procedure is simpler: Since $f(x)>0$ for all x, we can take logarithms in (1):

$$\ln\circ f(x+y)-\ln\circ f(x)+\ln\circ f(y).$$

Let $\ln\circ f=g$. Then $g(x+y)=g(x)+g(y)\Rightarrow g(x)=cx\Rightarrow \ln\circ f(x)=cx$, and

$$f(x)=e^{cx}.$$

E4. We treat the following equation more generally:

$$f(xy)=f(x)+f(y), \quad x,\ y>0. \tag{1}$$

We set $x=e^u,\ y=e^v,\ f(e^u)=g(u)$. Then (1) is transformed into $g(u+v)=g(u)+g(v)$ with solution $g(u)=cu$, and $f(x)=c\ \ln x$, as in **E1**, where we used differentiability.

E5. Next we consider the last Cauchy equation

$$f(xy)=f(x)f(y). \tag{1}$$

We assume $x > 0$ and $y > 0$. Then we set $x = e^u$, $y = e^v$, $f(e^u) = g(u)$ and get $g(u + v) = g(u) + g(v)$ with the solution $g(u) = e^{cu} = (e^u)^c = x^c$.

$$f(x) = x^c$$

and with the trivial solution $f(x) = 0$ for all x.

If we require (1) for all $x \neq 0,\ y \neq 0$, then $x = y = t$ and $x = y = -t$ give

$$f^2(t) = f(t^2) = f(-t)f(-t)$$

and

$$f(\ \ t) = \begin{cases} f(t) = t^c & \text{(or 0)}, \\ -f(t) = -t^c. \end{cases}$$

In this case the general continuous solutions are

$$(a) \quad f(x) = |x|^c, \qquad (b) \quad f(x) = \operatorname{sgn} x \cdot |x|^c, \qquad (c) \quad f(x) = 0.$$

E6. Now we come to Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}. \tag{1}$$

We set $f(0) = a$ and $y = 0$ and get $f\left(\frac{x}{2}\right) = \frac{f(x)+a}{2}$. Then

$$\frac{f(x) + f(y)}{2} = f\left(\frac{x+y}{2}\right) = \frac{f(x+y) + a}{2},$$
$$f(x + y) = f(x) + f(y) - a.$$

With $g(x) = f(x) - a$, we get $g(x + y) = g(x) + g(y),\ g(x) = cx$, and

$$f(x) = cx + a.$$

E7. Now we come to our last and most complicated example

$$f(x + y) + f(x - y) = 2f(x)f(y). \tag{1}$$

We want to find the continuous solutions of (1). First we eliminate the trivial solution $f(x) = 0$ for all x. Now

$$y = 0 \Rightarrow 2f(x) = 2f(x)f(0) \Rightarrow f(0) = 1,$$
$$x = 0 \Rightarrow f(y) + f(-y) = 2f(0)f(y) \Rightarrow f(-y) = f(y),$$

that is, f is an even function. For $x = ny$, we get

$$f\left[(n + 1)y\right] = 2f(y)f(ny) - f\left[(n - 1)y\right]. \tag{2}$$

For $y = x$, we get $f(2x) + f(0) = 2f^2(x)$. From this we conclude with $t = 2x$ that

$$f^2\left(\frac{t}{2}\right) = \frac{f(t) + 1}{2}. \tag{3}$$

(2) and (3) are satisfied by the functions cos and cosh. Since $f(0) = 1$ and f is continuous, we have $f(x) > 0$ in $[-a, a]$ for sufficiently small $a > 0$. Thus, $f(a) > 0$.

(a) *First case.* $0 < f(a) \leq 1$. Then there will be a c from $0 \leq c \leq \frac{\pi}{2}$, so that $f(a) = \cos c$. We show that, for any number of the form $x = (n/2^m)a$,

$$f(x) = \cos \frac{c}{a}x. \tag{4}$$

For $x = a$, this is valid by definition of c. Because of (3), for $x = a/2$,

$$f^2\left(\frac{a}{2}\right) = \frac{f(a) + 1}{2} = \frac{\cos c + 1}{2} = \cos^2 \frac{c}{2}.$$

Because of $f(a/2) > 0$, $\cos \frac{c}{2} > 0$, we conclude that

$$f\left(\frac{a}{2}\right) = \cos \frac{c}{2}. \tag{5}$$

Suppose (5) is valid for $x = a/2^m$. Then (3) implies

$$f^2\left(\frac{a}{2^{m+1}}\right) = \frac{f\left(\frac{a}{2^m}\right) + 1}{2} = \cos^2 \frac{c}{2^{m+1}}$$

or

$$f\left(\frac{a}{2^{m+1}}\right) = \cos \frac{c}{2^{m+1}},$$

that is, $f\left(a/2^m\right) = \cos\left(c/2^m\right)$ for every natural number m. Because of (2) for $n = 2$,

$$\begin{aligned} f\left(\frac{3}{2^m}\, a\right) &= f\left(3 \cdot \frac{a}{2^m}\right) = 2f\left(\frac{a}{2^m}\right) f\left(\frac{a}{2^{m-1}}\right) - f\left(\frac{a}{2^m}\right) \\ &= 2\cos \frac{c}{2^m} \cos \frac{c}{2^{m-1}} - \cos \frac{c}{2^m} = \cos \frac{3}{2^m}\, c. \end{aligned}$$

Since (4) is valid for $x = [(n-1)/2^m]a$ and $x = (n/2^m)a$, we conclude from (2) for $x = [(n-1)/2^m]a$ and $x = (n/2^m)a$, that

$$f\left(\frac{n+1}{2^m} a\right) = \cos \frac{n+1}{2^m}\, c.$$

Hence, we have

$$f\left(\frac{n}{2^m}\, a\right) = \cos \frac{n}{2^m}\, c \quad \text{for } n,\ m \in \{0, 1, 2, 3, \ldots\}.$$

Since f is continuous and even, we have

$$f(x) = \cos \frac{c}{a} x \quad \text{for all } x.$$

Second case. If $f(a) > 1$, then there is a $c > 0$, so that

$$f(a) = \cosh c.$$

One can show exactly as in the first case that

$$f(x) = \cosh \frac{c}{a} x \quad \text{for all } x.$$

Thus, the functional equation (1) has the following continuous solutions:

$$f(x) = 0, \quad f(x) = \cos bx, \quad f(x) = \cosh bx.$$

This list also contains $f(x) = 1$ for $b = 0$.

(b) We want to find all differentiable solutions of (1). Since differentiability is a far more powerful property than continuity, it will be quite easy to find all solutions of $f(x+y) + f(x-y) = 2f(x)f(y)$. We differentiate twice with respect to each variable:

With respect to x: $f''(x+y) + f''(x-y) = 2f''(x)f(y)$.
With respect to y: $f''(x+y) + f''(x-y) = 2f(x)f''(y)$.

From both equations we conclude that

$$f''(x) \cdot f(y) = f(x) \cdot f''(y) \Rightarrow \frac{f''(x)}{f(x)} = \frac{f''(y)}{f(y)} = c \Rightarrow f''(x) = cf(x),$$
$$c = -\omega^2 \Rightarrow f(x) = a \cos \omega x + b \sin \omega x,$$
$$c = \omega^2 \Rightarrow f(x) = a \cosh \omega x + b \sinh \omega x.$$

$f(0) = 1$ and $f(-x) = f(x)$ result in $f(x) = \cos \omega x$ and $f(x) = \cosh \omega x$, respectively.

Problems

1. Find some (all) functions f with the property $f(x) = f\left(\frac{x}{2}\right)$ for all $x \in \mathbb{R}$.
2. Find all continuous solutions of $f(x+y) = g(x) + h(y)$.
3. Find all solutions of the functional equation $f(x+y) + f(x-y) = 2f(x)\cos y$.
4. The function f is periodic, if, for fixed a and any x,

$$f(x+a) = \frac{1 + f(x)}{1 - f(x)}.$$

5. Find all polynomials p satisfying $p(x+1) = p(x) + 2x + 1$.

6. Find all functions f which are defined for all $x \in \mathbb{R}$ and, for any x, y, satisfy

$$xf(y) + yf(x) = (x+y)f(x)f(y).$$

7. Find all real, not identically vanishing functions f with the property

$$f(x)f(y) = f(x-y) \quad \text{for all } x,\ y.$$

8. Find a function f defined for $x > 0$, so that $f(xy) = xf(y) + yf(x)$.
9. The rational function f has the property $f(x) = f(1/x)$. Show that f is a rational function of $x + 1/x$.

   Remark. A rational function is the quotient of two polynomials.
10. Find all "tame" solutions of $f(x+y) + f(x-y) = 2\,[f(x) + f(y)]$.
11. Find all "tame" solutions of $f(x+y) - f(x-y) = 2f(y)$.
12. Find all "tame" solutions of $f(x+y) + f(x-y) = 2f(x)$.
13. Find all tame solutions of

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}.$$

14. Find all tame solutions of $f^2(x) = f(x+y)f(x-y)$. Note the similarity to 11.
15. Find the function f which satisfies the functional equation

$$f(x) + f\left(\frac{1}{1-x}\right) = x \quad \text{for all } x \neq 0,\ 1.$$

16. Find all continuous solutions of $f(x-y) = f(x)f(y) + g(x)g(y)$.
17. Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f^2(x)}$$

   holds for all x.

   (a) Prove that the function f is periodic, i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x.

   (b) For $a = 1$, give an example of a nonconstant function with the required properties (IMO 1968).
18. Find all continuous functions satisfying $f(x+y)f(x-y) = [f(x)f(y)]^2$.
19. Let $f(n)$ be a function defined on the set of all positive integers and with all its values in the same set. Prove that if

$$f(n+1) > f\,[f(n)]$$

   for each positive integer n, then $f(n) = n$ for each n (IMO 1977).
20. Find all continuous functions in 0 which satisfy the relations

$$f(x+y) = f(x) + f(y) + xy(x+y), \quad x,\ y \in \mathbb{R}.$$

21. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

$$\text{(i)} \quad f\,[xf(y)] = yf(x) \quad \text{for all positive } x,\ y;$$
$$\text{(ii)} \quad f(x) \to 0 \quad \text{as} \quad x \to \infty \quad \text{(IMO 1983)}.$$

22. Find all functions f, defined on the nonnegative real numbers and taking nonnegative real values, such that

$$\text{(i)} \quad f[xf(y)]\,f(y) = f(x+y) \quad \text{for all } x,\ y \geq 0;$$
$$\text{(ii)} \quad f(2) = 0;$$
$$\text{(iii)} \quad f(x) \neq 0 \quad \text{for } 0 < x < 2 \quad \text{(IMO 1986)}.$$

23. Find a function $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^-$, which satisfies, for all $x,\ y \in \mathbb{Q}^+$, the equation

$$f\,(xf\,(y)) = f\,(x)\,/\,y \quad \text{(IMO 1990)}.$$

24. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f\left[x^2 + f\,(y)\right] = y + [f\,(x)]^2 \quad \text{for all } x,\ y \in \mathbb{R} \quad \text{(IMO 1992)}.$$

25. Does there exist a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$f\,(1) = 2,\ \ f\,[f\,(n)] = f\,(n) + n,\ \ f\,(n) < f\,(n+1) \quad \text{for all } n \in \mathbb{N} \ \text{(IMO 1993)?}$$

26. Find all continuous functions $f : \mathbb{R} \mapsto \mathbb{R}_+$ which transform three terms of the arithmetic progression $x,\ x-y,\ x+2y$ into corresponding terms $f(x),\ f(x+y),\ f(x+2y)$ of a geometric progression, that is,

$$[f(x+y)]^2 = f(x) \cdot f(x+2y).$$

27. Find all continuous functions f satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$.

28. Guess a simple function f satisfying $f^2(x) = 1 + xf(x+1)$.

29. Find all continuous functions which transform three terms of an arithmetic progression into three terms of an arithmetic progression.

30. Find all continuous functions f satisfying $3f(2x+1) = f(x) + 5x$.

31. Which function is characterized by the equation $xf(x) + 2xf(-x) = -1$?

32. Find the class of continuous functions satisfying $f(x+y) = f(x) + f(y) + xy$.

33. Let $a \neq \pm 1$. Solve $f\,(x/(x-1)) = af(x) + \phi(x)$, where $\phi(x)$ is a given function, which is defined for $x \neq 1$.

34. The function f is defined on the set of positive integers as follows:

$$f(1) = 1, \qquad f(3) = 3, \qquad f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n), \qquad f(4n+3) = 3f(2n+1) - 2f(n).$$

Find all values of n with $f(n) = n$ and $1 \leq n \leq 1988$ (IMO 1988).

35. A function f is defined on the set of rational numbers as follows:

$$f(0)=0, \quad f(1)=1, \quad f(x)=\begin{cases} f(2x)/4 & \text{for } 0<x<\frac{1}{2}, \\ \frac{3}{4}+f(2x-1)/4 & \text{for } \frac{1}{2}\le x<1. \end{cases}$$

Let $a=0.b_1b_2b_3\cdots$ be the binary representation of a. Find $f(a)$.

36. Find all polynomials over $\mathbb{C}$ satisfying $f(x)f(-x)=f(x^2)$.

37. The strictly increasing function $f(n)$ is defined on the positive integers and it assumes positive integral values for all $n\ge 1$. In addition, it satisfies the condition $f[f(n)]=3\cdot n$. Find $f(1994)$ (IIM 1994).

38. (a) The function $f(x)$ is defined for all $x>0$ and satisfies the conditions

 (1) $f(x)$ is strictly increasing on $(0,+\infty)$,

 (2) $f(x)>-1/x$ for $x>0$,

 (3) $f(x)\cdot f(f(x)+1/x)=1$ for all $x>0$.

 Find $f(1)$.

 (b) Give an example of a function $f(x)$ which satisfies (a).

39. Find all sequences $f(n)$ of positive integers satisfying

$$f[f[f(n)]]+f[f(n)]+f(n)=3n.$$

40. Find all functions $f:\mathbb{N}_0\mapsto\mathbb{N}_0$, such that

$$f[m+f(n)]=f[f(m)]+f(n) \quad \text{for all } m,n\in\mathbb{N}_0 \quad \text{(IMO 1996)}.$$

Solutions

1. Any constant function has the required property. Another example is the function f defined by $f(x)=|x|/x, \quad x\ne 0$. For 0, one can define f arbitrarily.

 There are infinitely many solutions. One can get all solutions as follows: Take any interval of the form $[a,2a]$. For instance, let us take $[1,2]$. Define f in this interval, arbitrarily, except $f(1)=f(2)$. Then f is defined for all real $x>0$. Take the graph of f in $[1,2]$, and stretch it horizontally by the factor 2^n (n an integer). Then you get the graph of f in the interval $[2^n,2^{n+1}]$. We can define $f(0)$ as we please. For negative x we can again choose an interval $[b,2b]$, $b<0$, define f in this interval arbitrarily except $f(b)=f(2b)$, and extend the definition to all negative x by stretching it.

2. This equation can be reduced to Cauchy's equation. Set $y=0$, $h(0)=b$. You get

$$f(x)=g(x)+b, \qquad g(x)=f(x)-b.$$

 For $x=0$, $g(0)=a$ we get $f(y)=a+h(y)$, $\quad h(y)=f(y)-a$. Thus, $f(x+y)=f(x)+f(y)-a-b$. So with $f_0(z)=f(z)-a-b$, we have

$$f_0(x+y)=f_0(x)+f_0(y),$$

 i.e., $f_0(x)=cx$, and

$$f(x)=cx+a+b, \quad g(x)=cx+a, \quad h(x)=cx+b.$$

3. For $y = \pi/2$, the right side disappears. We substitute $x = 0$, $y = t$, $x = \frac{\pi}{2} + t$, $y = \frac{\pi}{2}$, $x = \frac{\pi}{2}$, $y = \frac{\pi}{2} + t$, and we get

$$f(t) + f(-t) = 2a \cos t, \quad f(\pi + t) + f(t) = 0, \quad f(\pi + t) + f(-t) = -2b \sin t,$$

where $a = f(0),\ b = f\left(\frac{\pi}{2}\right)$. Hence,

$$f(t) = a \cos t + b \sin t.$$

4. We find that $f(x + 2a) = -1/f(x)$, i.e., $f(x + 4a) = f(x)$. Thus $4a$ is a period of f.

5. We can guess the solution $p(x) = x^2$. Is it the only one? A standard method for answering this question is to introduce the difference $f(x) = p(x) - x^2$. The given functional equation becomes $f(x + 1) = f(x)$. So $f(x) = c$, a constant. Thus $p(x) = x^2 + c$. We **must** check if this solution satisfies the original equation, which is indeed the case.

6. $y = x \Rightarrow f(x) = f^2(x) \Rightarrow f(x)(f(x) - 1) = 0$ for all x. Continuous solutions are $f(x) \equiv 0, \quad f(x) \equiv 1$. There are many more discontinuous solutions. On any subset A of $\mathbb{R}$, set $f(x) = 0$. On $\mathbb{R} \setminus A$, set $f(x) = 1$. But there is a restriction, which we find by setting $y = -x$. It shows that $f(-x) = f(x)$ for all x, i.e., f is an even function.

7. $y = 0 \Rightarrow f(x)f(0) = f(x)$ for all x. Since f is not identically vanishing, we must have $f(0) = 1$. $y = x \Rightarrow f(x)f(x) = 1$ for all x. We get two continuous functions $f(x) \equiv 1$ and $f(x) \equiv -1$. There are many discontinuous functions, e.g., $f(x) = 1$ on any subset A of $\mathbb{R}$, and $f(x) = -1$ on $\mathbb{R} \setminus A$.

8. Let $g(x) = (f(x))/x$. Then we get the Cauchy equation $g(xy) = g(x) + g(y)$ with the solution $g(x) = c \ln x$. This implies $f(x) = cx \ln x$.

9. Suppose

$$f(x) = \frac{x^k(a_0 x^n + a_1 x^{n-1} + \cdots + a_n)}{x^l(b_0 x^m + \cdots + b_m)},$$

where $a_0,\ b_0,\ a_n,\ b_m$ are not zero. Using the relation $f(x) = f(1/x)$, we get

$$\frac{x^{2(l-k)+m-n}(a_n x^n + \cdots + a_0)}{(b_m x^m + \cdots + b_0)} \equiv \frac{a_0 x^n + \cdots + a_n}{b_0 x^m + \cdots + b_m}. \tag{1}$$

From here we get $m - n = 2(k - l)$, where m and n have the same parity. From (1) we conclude that

$$P_m(x) = b_m x^m + \cdots + b_0 = b_0 x^m + \cdots + b_m$$

and

$$P_n(x) - a_n x^n + \cdots + a_0 \equiv a_0 x^n + \cdots + a_n,$$

i.e., $a_0 = a_n, a_1 = a_{n-1}, \cdots;\ b_0 = b_m,\ b_1 = b_{m-1}, \ldots$. Hence $P_m(x)$ and $P_n(x)$ are reciprocal polynomials, which can be represented as follows: For even n: $n = 2r$, then $P_{2r}(x) = x^n g_r(z)$, where $z = x - 1/x$ and $g(z)$ is a polynomial of degree r. If n is odd: $n = 2r + 1$, then $P_{2r+1}(x) - (x + 1)x^r h_r(z)$, where $z - x + 1/x$, and $h_r(z)$ is a polynomial of degree r.

Furthermore, there are two possibilities:

(a) $m = 2s,\ n = 2r$. Then

$$f(x) = \frac{x^k x^r g_r(z)}{x^l x^s h_s(z)} = \frac{g(z)}{h(z)}.$$

(b) $m = 2s + 1,\ n = 2r + 1$. Then

$$f(x) = \frac{(x+1)x^{k+r} g_r(z)}{(x+1)x^{l+s} h_s(z)} = \frac{g(z)}{h(z)}.$$

10. For $y = 0$, we get $2f(x) = 2f(x) + 2f(0)$, or $f(0) = 0$. For $x = y$, we have $f(2x) = 4f(x)$. We prove by induction that $f(nx) = n^2 f(x)$ for all x. Now let $x = p/q$. Then $qx = p \cdot 1$, $f(qr) = f(p \cdot 1)$, $q^2 f(x) = p^2 f(1)$. With $f(1) = a$, we get $f(x) = ax^2$ for all rational x. By continuity we can extend this to all continuous functions. By putting $f(x) = ax^2$ into the original equation, we see that it is indeed satisfied.

11. For $y = 0$, we get $f(x) - f(x) = 2f(0)$, or $f(0) = 0$. For $y = x$, we get $f(2x) = 2f(x)$ for all x. By induction we prove that $f(nx) = nf(x)$. Now let $x = p/q$ or $qx = p \cdot 1$. Then $f(qx) = f(p \cdot 1) \Rightarrow qf(x) = pf(1) \Rightarrow f(x) = f(1)x$ for all rational x. By continuity this can be extended to all real x. Putting $f(x) = ax$ into the functional equation, we see that it is the solution.

12. We want to solve the functional equation $f(x + y) + f(x - y) = 2f(x)$. $y = x$ yields $f(2x) + f(0) = 2f(x)$, or $f(2x) = 2f(x) + b$ with $b = -f(0)$. Now $f(2x + x) + f(2x - x) = 2f(2x)$ yields $f(3x) + f(x) = 2(2f(x) + b)$, or $f(3x) = 3f(x) + 2b$. We guess $f(nx) = nf(x) + (n-1)b$, and we prove this by induction. Now let $x = p/q \Leftrightarrow qx = p \cdot 1$ with $p,\ q \in \mathbb{N}$. Then $f(qx) = f(p \cdot 1)$, or $qf(x) + (q - 1)b = pf(1) + (p-1)b$, or $f(x) = f(1)x + (x-1)b$, or $f(x) = \lfloor f(0) + f(1) \rfloor x - b$. With $f(0) + f(1) = a$ and $f(0) = b$, we finally get $f(x) = ax + b$. A check shows that this is indeed a solution.

13. Setting $g(x) = 1/f(x)$, we get Cauchy's equation $g(x + y) = g(x) + g(y)$ with the solution $g(x) = cx$. Thus $f(x) = 1/cx$ is the general continuous solution.

14. Taking logarithms on both sides, we get $2g(x) = g(x + y) + g(x - y)$. Here $g(x) = ln \circ f(x)$, that is, $g(x) = ax - b$. Thus $f(x) = e^{ax+b}$, or $f(x) = rs^x$.

15. We repeatedly replace $x \xrightarrow{g} 1/(1 - x)$ and get

$$x \xrightarrow{g} \frac{1}{1-x} \xrightarrow{g} 1 - \frac{1}{x} \xrightarrow{g} x.$$

We get the following equations:

$$f(x) + f\left(\frac{1}{1-x}\right) = x,\ f\left(\frac{1}{1-x}\right) + f\left(1 - \frac{1}{x}\right) = \frac{1}{1-x},\ f\left(1 - \frac{1}{x}\right) + f(x) = 1 - \frac{1}{x}.$$

Eliminating $f\left(\frac{1}{1-x}\right)$ and $f\left(1 - \frac{1}{x}\right)$ we get $f(x) = \frac{1}{2}\left(1 + x - \frac{1}{x} - \frac{1}{1-x}\right)$. A check shows that this function indeed satisfies the functional equation.

16. *Hint*: Interchanging x with y, we see that $f(-x) = f(x)$ for all x. Setting $y = 0$, we get $f(0)^2 = f^2(x) + g^2(x)$. $x = y = 0$ implies $f(0) = f^2(0) + g^2(0)$. $y = 0$ implies $f(x) = f(x)f(0) + g(x)g(0)$. Now $f(0) = 0$ would imply $g(0) = 0$ and $f(x) \equiv 0$ for all x. Thus, $f(0) \neq 0$. But $f(x)[1 - f(0)] = g(x)g(0)$. Thus, $f(0) = 1$, and hence $g(0) = 0$. $y = -x$ implies $f(2x) = f^2(x) + g(x)g(-x)$. We should get $f(x) = \cos x$ and $g(x) = \sin x$.

17. We have $f(x+a) \geq \frac{1}{2}$, and so $f(x) \geq \frac{1}{2}$ for all x. If we set $g(x) = f(x) - \frac{1}{2}$, we have $g(x) \geq 0$ for all x. The given functional equation now becomes
$$g(x+a) = \sqrt{\frac{1}{4} - [g(x)]^2}.$$
Squaring, we get
$$[g(x+a)]^2 = \frac{1}{4} - [g(x)]^2 \text{ for all } x, \tag{1}$$
and thus also
$$[g(x+2a)]^2 = \frac{1}{4} - [g(x+a)]^2.$$
These two equations imply $[g(x+2a)]^2 = [g(x)]^2$. Since $g(x) \geq 0$ for all x, we can take square roots to get $g(x+2a) = g(x)$, or
$$f(x+2a) - \frac{1}{2} = f(x) - \frac{1}{2},$$
and
$$f(x+2a) = f(x) \quad \text{for all } x.$$
This shows that $f(x)$ is periodic with period $2a$.
(b) To find all solutions, we set $h(x) = 4[g(x)]^2 - \frac{1}{2}$. Now (1) becomes
$$h(x-a) = -h(x). \tag{2}$$
Conversely, if $h(x) \geq \frac{1}{2}$ and satisfies (2), then $g(x)$ satisfies (1). An example for $a = 1$ is furnished by the function $h(x) = \sin^2 \frac{\pi}{2}x - \frac{1}{2}$ which satisfies (2) with $a = 1$. For this h, $g(x) = \frac{1}{2}|\sin(\pi x/2)|$ and
$$f(x) = \frac{1}{2}\left|\sin \frac{\pi}{2}x\right| + \frac{1}{2}.$$
In fact, $h(x)$ can be defined arbitrarily in $0 \leq x < a$ subject to the condition $|h(x)| \leq \frac{1}{2}$ and extended to all x by (2).

18. To find the solution of $f(x-y)f(x+y) = [f(x)f(y)]^2$, we observe that we can assume f to be nonnegative. In fact, all we can say about a positive f is also valid for a negative f. The three trivial solutions $f(x) \equiv 0,\ 1,\ -1$ will be excluded from now on. $y = 0 \Rightarrow f(x)^2 = f(x)^2 f(0)^2 \Rightarrow f(0)^2 = 1 \Rightarrow f(0) = 1$.
$x = 0 \Rightarrow f(y)f(-y) = f(y)^2 \Rightarrow f(y) = f(-y)$. Thus, f is an even function.
$x = y \Rightarrow f(2x) = f(x)^4$. By induction we get $f(nx) = f(x)^{n^2}$. This can be extended to rationals and then reals as in **E2**. Finally, we get
$$f(x) = f(1)^{x^2} \quad \text{for all } x.$$
Another approach introduces $g = \ln \circ f$ to get $g(x+y) + g(x-y) = 2\,(g(x) + g(y))$. This suggests the identity $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$. Thus we guess $g(x) = ax^2$ and $f(x) = e^{ax^2}$. It remains to be proved that the guess is unique.

19. f has a unique minimum at $n = 1$. For, if $n > 1$, we have $f(n) > f[f(n-1)]$. By the same reasoning, we see that the second smallest value is $f(2)$, etc. Hence,

$$f(1) < f(2) < f(3) < \cdots.$$

Since $f(n) \geq 1$ for all n, we also have $f(n) \geq n$. Suppose that, for some positive integer k, we have $f(k) > k$. Then $f(k) \geq k+1$. Since f is increasing, $f(f(k)) \geq f(k+1)$, contradicting the given inequality. Hence $f(n) = n$ for all n.

20. It is easy to guess the solution from this property. The function $x^3/3$ satisfies the relationship. So we consider $g(x) = f(x) - x^3/3$. For g we get the functional equation $g(x+y) = g(x) + g(y)$. Since $g(x) = cx$ is the only continuous solution in 0, we have $f(x) = cx + x^3/3$.

21. We show that 1 is in the range of f. For an arbitrary $x_0 > 0$, let $y_0 = 1/f(x_0)$. Then (i) yields $f[x_0 f(y_0)] = 1$, so 1 is in the range of f. In the same way, we can show that any positive real is in the range of f. Hence there is a value y such that $f(y) = 1$. Together with $x = 1$ in (i), this gives $f(1 \cdot 1) = f(1) = yf(1)$. Since $f(1) > 0$ by hypothesis, it follows that $y = 1$, and $f(1) = 1$. We set $y = x$ in (i) and get

$$f[xf(x)] = xf(x) \quad \text{for all } x > 0. \tag{1}$$

Hence, $xf(x)$ is a fixed point of f. If a and b are fixed points of f, that is, if $f(a) = a$ and $f(b) = b$, then (i) with $x = a,\ y = b$ implies that $f(ab) = ba$, so ab is also a fixed point of f. Thus the set of fixed points of f is closed under multiplication. In particular, if a is a fixed point, all nonnegative integral powers of a are fixed points. Since $f(x) \to 0$ for $x \to \infty$ by (ii), there can be no fixed points > 1. Since $xf(x)$ is a fixed point, follows that

$$xf(x) \leq 1 \Leftrightarrow f(x) \leq \frac{1}{x} \quad \text{for all } x. \tag{2}$$

Let $a = zf(z)$, so $f(a) = a$. Now set $x = 1/a$ and $y = a$ in (i) to give

$$f\left[\frac{1}{a}f(a)\right] = f(1) = 1 = af\left(\frac{1}{a}\right), \quad f\left(\frac{1}{a}\right) = \frac{1}{a}, \quad f\left[\frac{1}{zf(z)}\right] = \frac{1}{zf(z)}.$$

This shows that $1/xf(x)$ is also a fixed point of f for all $x > 0$. Thus, $f(x) \geq 1/x$. Together with (2) this implies that

$$f(x) = \frac{1}{x}. \tag{3}$$

The function (3) is the only solution satisfying the hypothesis.

22. No solution.

23. If $f(y_1) = f(y_2)$, the functional equation implies that $y_1 = y_2$. For $y = 1$, we get $f(1) = 1$. For $x = 1$, we get $f(f(y)) = \frac{1}{y}$ for all $y \in \mathbb{Q}^+$. Applying f to this implies that $f(1/y) = 1/f(y)$ for all $y \in \mathbb{Q}^+$. Finally setting $y = f(1/t)$ yields $f(xt) = f(x) \cdot f(t)$ for all $x,\ t \in \mathbb{Q}^+$.

Conversely, it is easy to see that any f satisfying

$$\text{(a) } f(xt) = f(x)f(t), \quad \text{(b)} \quad f[f(x)] = 1/x \quad \text{for all } x,\ t \in \mathbb{Q}^+$$

solves the functional equation.

A function $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ satisfying (a) can be constructed by defining arbitrarily on prime numbers and extending as

$$f\left(p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}\right) = [f(p_1)]^{n_1} [f(p_2)]^{n_2} \cdots [f(p_k)]^{n_k},$$

where p_j denotes the jth prime and $n_j \in \mathbb{Z}$. Such a function will satisfy (b) for each prime.

A possible construction is as follows:

$$f(p_j) = \begin{cases} p_{j+1} & \text{if } j \text{ is odd,} \\ \frac{1}{p_{j-1}} & \text{if } j \text{ is even.} \end{cases}$$

Extending it as above, we get a function $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$. Clearly $f[f(p)] = 1/p$ for each prime p. Hence f satisfies the functional equation.

24. No solution.

25. Starting with $f(1) = 2$ and using the rule $f[f(n)] = f(n) + n$, we get, successively, $f(2) = 2 + 1 = 3$, $f(3) = 3 + 2 = 5$, $f(5) = 5 + 3 = 8$, $f(8) = 8 + 5 = 13, \ldots$ that is, the map of a Fibonacci number is the next Fibonacci number. Complete this by induction.

It remains to assign other positive integers to the remaining numbers satisfying the functional equation. We use Zeckendorf's theorem, which says that every positive integer n has a unique representation as a sum of non-neighboring Fibonacci numbers. We have proved this in Chapter 8, problem 29. We write this representation in the form

$$n = \sum_{j=1}^{m} F_{i_j}, \quad |i_j - i_{j-1}| \geq 2,$$

where the summands have increasing indices. We will prove that the function $f(n) = \sum_{j=1}^{m} F_{i_j+1}$ satisfies all conditions of the problem. Indeed, since 1 represents itself as a Fibonacci number, we have $f(1) = 2$, the next Fibonacci number. Then

$$f[f(n)] = f\left(\sum_{i_j+1}^{m} F_{i_j+1}\right) = \sum_{j=1}^{m} F_{i_1+2} = \sum_{j=1}^{m} \left(F_{i_j+1} - F_{i_j}\right)$$
$$= \sum_{j=1}^{m} F_{i_j+1} + \sum_{j=1}^{m} F_{i_j} = f(n) + n.$$

Now we distinguish two cases.

(a) The Fibonacci representation of n contains neither F_1 nor F_2. Then the representation of $n + 1$ contains the additional summand 1. The representations of $f(n)$ and $f(n+1)$ differ also by an additional summand in $f(n+1)$, so that $f(n) < f(n+1)$.

(b) The Fibonacci representation of n contains either F_1 or F_2. On adding of 1, some summands will become bigger Fibonacci numbers. The representation of $n+1$ has a largest Fibonacci number which is larger than the largest Fibonacci representation of n. This property remains invariant after the application of f. Hence $f(n+1) > f(n)$, since the summands in the representation of $f(n)$ are nonneighboring Fibonacci numbers and cannot add up to the greatest Fibonacci number in $f(n+1)$.

Remark. The function f is not uniquely determined by the three conditions.

26. Replacing $x \to x - y$, we get the equation

$$f(x)^2 = f(x-y)f(x+y).$$

We can assume that f is positive. By introducing $g = \ln \circ f$, we get

$$g(x-y) + g(x+y) = 2g(x),$$

which we solved in problem 13. A similar one was solved in 11.

27. By setting $f(x) = g(x) - 1$, we can radically simplify the functional equation

$$g(x+y) = g(x)g(y).$$

This is the functional equation of the exponential function $g(x) = a^x$, or

$$f(x) = a^x - 1.$$

28. The only solution is $f(x) = x + 1$. See [21], problem 18.
29. We must solve the equation $f(x) + f(x+2y) = 2f(x+y)$. The result is $f(x) = ax + b$.
30. The unique solution is $f(x) = x - \frac{3}{2}$. Show this yourself.
31. We replace x by $-x$ and get $-xf(-x) - 2xf(x) = -1$. Thus, we have two equations for $f(x)$ and $f(-x)$. Solving for $f(x)$, we get $f(x) = 1/x$.
32. We guess $f(x) = ax^2 + bx + c$. Inserting this guess into the equation, we get $a(x+y)^2 - ax^2 + ay^2 + xy$, or $ax^2 + ay^2 + 2axy + b(x+y) + c - ax^2 + bx + c + ay^2 + by + c + xy$, which is satisfied for $a = 1/2$ and $c = 0$. By more conventional methods, show that $f(x) = x^2/2 + c$ is the only continuous solution.
33. Let $y = \frac{x}{x-1}$. Then $x = \frac{y}{y-1}$. Thus $f(y) = (a\phi(y) + \phi(y/y-1))/(1-a^2)$.
34. Any positive integer n can be written in the binary system, e.g., $1988 = 11111000100_2$. By induction on the number in the binary system, we will prove the following assertion: if

$$n = a_0 2^k + a_1 2^{k-1} + \cdots + a_k, \quad a_0, \ldots, a_k \in \{0, 1\},\ a_0 = 1,$$

then

$$f(n) = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_0.$$

For $1 = 1_2,\ 2 = 10_2,\ 3 = 11_2$ the assertion is true because of the first three points in (1). Now, suppose that the assertion is true for all numbers with less than $(k+1)$ digits in the binary system. Let

$$n = a_0 2^k + a_1 2^{k-1} + \cdots + a_k, \quad a_0 = 1.$$

We consider three cases: (a) $a_k = 0$, (b) $a_k = 1$, $a_{k-1} = 0$ and (c) $a_k = a_{k-1} = 1$. We only consider the case (b), the remaining cases can be handled similarly. In case (b) $n = 4m + 1$, where

$$m = a_0 2^{k-2} + \cdots + a_{k-2}, \quad 2m + 1 = a_0 2^{k-1} + \cdots + a_{k-2} 2 + 1.$$

Because of (4), we have $f(n) = 2f(2m+1) - f(m)$. By the induction hypothesis

$$f(m) = a_{k-2} 2^{k-2} + \cdots + a_0, \quad f(2m+1) = 2^{k-1} + a_{k-2} 2^{k-2}.$$

Hence,

$$f(n) = 2^k + 2(a_{k-2}2^{k-2} + \cdots + a_0) - (a_{k-2}2^{k-2} + \cdots + a_0)$$
$$= 2^k + a_{k-2}2^{k-2} + \cdots + a_0 = a_k 2^k + a_{k-1}2^{k-1} + \cdots + a_0,$$

q.e.d. The problem was to find the number of integers ≤ 1988 with symmetric binary representation. We observe that this number is $2^{\lfloor (n-1)/2 \rfloor}$. We also see that only two symmetric 11-digit numbers 11111111111_2 and 11111011111_2 are larger than 1988. Hence the number we are seeking is

$$(1+1+2+2+2^2+2^2+\cdots+2^4+2^4+2^5)-2 = (2^5-1)+(2^6-1)-2 = 92.$$

35. Let $x = 0.b_1b_2b_3\cdots$. If $b_1 = 0$, then $x < \frac{1}{2}$ and $f(x) = 0.b_1b_1 + \frac{1}{4}f(0.b_2b_3\cdots)$. If $b_1 = 1$, then $x \geq \frac{1}{2}$, and $f(x) = 0.b_1b_1 + \frac{1}{4}f(0.b_2b_3\cdots)$. From this we conclude that $f(x) = 0.b_1b_1b_2b_2b_3b_3\cdots$.

36. If z is a root of f, then also z^2 is. If $|z| \neq 1$, there are infinitely many roots, which is a contradiction. Hence all roots lie at the origin or *on the unit circle*. 0, 1 and third roots of unity have the closure property for squaring. Hence $x^p(x-1)^q(1+x+x^2)^r$ also has the closure property. Inserting into the functional equation, we see that, in addition, $p+q$ must be even:

$$f(x) = x^p(x-1)^q(1+x+x^2)^r, \quad p,\ q,\ r \in \mathbb{N}_0, \quad p+q \equiv 0 \bmod 2.$$

37. *Hint:* We have $f(1) < f(2) < f(3) < \cdots$. In addition we have $f(1) < f[f(1)] = 3$. Thus $f(1) = 2$, $f(2) = 3$. Prove that $f(3n) = 3f(n)$. In fact, $f(n) = n + 3^k$ for $3^k \leq n < 2\cdot 3^k$, and $f(n) = 3n - 3^{k+1}$ for $2\cdot 3^k \leq n < 3^{k+1}$. Hence $f(1994) = 3795$.

38. (a) Let $f(1) = t$. For $x = 1$, we have $tf(t+1) = 1$ and $f(t+1) = 1/t$. Now $x = t+1$ yields

$$f(t+1)f\left[f(t+1)+\frac{1}{t+1}\right] = 1 \Rightarrow f\left(\frac{1}{t}+\frac{1}{t+1}\right)$$
$$= t \Rightarrow f\left(\frac{1}{t}+\frac{1}{t+1}\right) = f(1).$$

Since f is increasing, we have $1/t + 1/(t+1) = 1$, or $t = (1 \pm \sqrt{5})/2$. But if t were positive, we would have the contradiction $1 < t = f(1) < f(1+t) = 1/t < 1$. Hence $t = (1-\sqrt{5})/2$ is the only possibility.

(b) Similar to the computation of $f(1)$, we can prove that $f(x) = t/x$, where $t = (1-\sqrt{5})/2$. Again we must check that this function indeed satisfies all conditions of the problem.

39. Obviously the sequence $f(n) = n$ satisfies the condition. We prove that there are no other solutions. We observe that the function f is injective. Indeed,

$$f(x) = f(y) \Rightarrow f[f(x)] = f[f(y)] \Rightarrow f\{f[f(x)]\} = f\{f[f(y)]\}$$
$$\Rightarrow f\{f[f(x)]\} + f[f(x)] + f(x) = f\{f[f(y)]\} + f[f(y)] + f(y)$$
$$\Rightarrow 3x = 3y,$$

which implies $x = y$. For $n = 1$, we easily get $f(1) = 1$. Suppose that, for $n < k$, we have $f(n) = n$. We prove that $f(k) = k$. If $p = f(k) < k$ then by the induction hypothesis $f(p) = p = f(k)$, and this contradicts the injectivity of f. If $f(k) > k$, then $f[f(k)] \geq k$. If we had $f[f(k)] < k$, then, as before, we would get the contradiction

$$f\{f[f(k)]\} = f[f(k)], \quad f[f(k)] = f(k), \quad f(k) = k.$$

Similarly, we have $f\{f[f(k)]\} \geq k$. Hence, $f\{f[f(k)]\} - f[f(k)] + f(k) > 3k$, which contradicts the original condition. Thus $f(k) = k$.