BAB V

PERSAMAAN DEFERENSIAL ORDE SATU DERAJAT SATU

Persamaan diferensial adalah persamaan yang berbentuk F (x, y, y', y'', y''', ..., y^n) = 0 yang menyatakan hubungan antara independent variabel x, derendent variabel y dan turunan-turunannya, y', y'', y''', ..., y^n .

Persamaan diferensial mempunyai orde (tingkat) dan derajad. Orde PD adalah turunan tertinggi yang ada di dalam persamaan diferensial, sedangkan derajad PD adalah pangkat pada turunan tertinggi dalam PD.

Contoh-contoh :

1. $x\frac{dy}{dx}$ + 5y = 6 ; orde PD : 1 ; derajad PD : 1
2. $\frac{d^3y}{dx^3}$ + 4 $(\frac{d^2y}{dx^2})^2 + \frac{dy}{dx}$ = sinx ; orde PD : 3 ; derajad PD : 1
3. $(\frac{d^3y}{dx^3})^2 + \frac{d^2y}{dx^2}$ + 2xy = 6; orde PD : 3 ; derajad PD : 2

A. PERSAMAAN DEFERENSIAL TERPISAH

Bentuk umum : $f(x)dx + g\,(y)\;dy = 0$

Solusi umum : $\int f(x)dx + \int g\,(y)\,dy = 0$

Contoh 1 :

Selesaikan PD berikut : x^5 dx + $(y + 2)^2$ dy = 0

Penyelesaian :

$\int x^5\,dx + \int (y + 2)^2\,dy = c_1$

➔ $\frac{1}{6}x^6 + 1/3\,(y + 2)^3 = c_1$

$X^6 + 2\,(y + 2)^3 = c$ ($c = 6c_1$)

Penyelesaian :

*) 9y dy + 4x dx = 0 ⟷ $\int 9y\,dy + \int 4x\,dx = 0_1$

⟷ $\frac{9}{2}y^2 + 2x^2 = c_1$

⟷ $\frac{y^2}{4} + \frac{x^2}{9} = c$ ($c = \frac{1}{18}c_1$)

B. REDUKSI KE PERSAMAAN DEFERENSIAL VARIABEL TERPISAH

Bentuk umum

$f_1(x)\ g_1(y)\,dx + f_2(x)\, g_2(y)\,dy = 0$

reduksi dengan faktor integral (FI) : $\frac{1}{g_1(y)\, f_2(x)}$, sehingga diperoleh :

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0$$

Penyelesaian umum :

$\int \frac{f_1(x)}{f_2(x)}$ dx + $\int \frac{g_2(y)}{g_1(y)}$ dy = c c = konstanta

Contoh 1 :

Selesaikan PD berikut : (1 + 2y) dx + (x – 4) dy = 0

Penyelesaian :

FI = $\frac{1}{(1+2y)(x-4)}$, maka

$\frac{1}{(1+2y)(x-4)}$ (1 + 2y) dx + (x – 4) dy = 0

➔ $\int \frac{1}{x-4}$ dx + $\int \frac{1}{(1+2y)}$ dy = 0

➔ $\int \frac{1}{x-4}$ dx + $\int \frac{1}{(1+2y)}$ dy = C_1

➔ ln / x-4/ + ½ ln/ 1+2y/ = C_1

➔ 2 ln/ x-4/ +ln/1+2y/ = C_2 ($C_2 = 2\ C_1$)

➔ ln $(/x-4/)^2$ + ln /1 + 2y/ = C_2

➔ <u>ln $(x-4)^2$ (1+2y) = C ($C = C_2^2$)</u>

Contoh 2 :

Selesaikan PD berikut : $\frac{dy}{dx} = \frac{4y}{xy-3x}$

Penyelesaian :

PD di atas dapat dituliskan sebagai :

$(xy - 3x)$ dy – 4y dx = 0 ⟺ x (y-3) dy – 4y dx = 0

FI : $\frac{1}{xy}$ diperoleh :

$\frac{1}{xy}$ [x (y-3) dy – 4y dx] = 0

⇨ $\frac{y-3}{y}$ dy - $\frac{4}{x}$ dx = 0

⇨ (1 - $\frac{3}{y}$) dy - $\frac{4}{x}$ dx = 0

⇨ $\int (1 - \frac{3}{y})\,dy - \int \frac{4}{x}\,dx = c_1$

⇨ y – 3 ln /y/ - 4 ln /x/ = c_1

⇨ y – ln $(y)^3$ – ln x^4 = c_1

⇨ y – ln y^3 x^4 = c_1

⇨ ln e^y – ln (y^3 x^4) = ln c (c_1 = ln c)

⇨ ln $\frac{e^y}{y^3\,x^4}$ = ln c

⇨ $\frac{e^y}{y^3\,x^4}$ = c ➔ $e^y = c\,y^3\,x^4$

C. PERSAMAAN DEFERENSIAL LINIER HOMOGEN

Bentuk umum

M (x, y) dx + N (x, y) dy = 0

Dengan : M (λ x, λ y) = λ^n M (x, y)

N (λ x, λ y) = λ^n N (x, y) dan n = derajad PD homogen.

Prosedur penyelesaian :

i. Transformasi : y = Ux ➔ dy = x du + udx atau x = uy ➔ dx = udy +ydu
ii. Reduksi PD ke PD variabel terpisah
iii. Selesaikan PD
iv. Gantikan u = $\frac{y}{x}$ bila transformasinya y = ux dan u = $\frac{x}{y}$ bila transformasinya x = uy untuk memperoleh variabel semula

Contoh 1.

Selesaikan PD berikut : 2x dy – 2y dx = $\sqrt{x^2 + 4y^2}$ dx

Penyelesaian :

PD di atas dapat dituliskan sebagai :

$(2y + \sqrt{x^2 + 4y^2}\ ) dx – 2x\ dy$

$M (x,y) = 2y + \sqrt{x^2 + 4y^2}$

➔ $M (\lambda x, \lambda y) \quad = 2 \lambda y + \lambda\sqrt{x^2 + 4y^2}$

$\qquad = \lambda (2y + \sqrt{x^2 + 4y^2}\ ) = \lambda M (x,y)$

$N (x,y) = -2x$

➔ $N (\lambda x, \lambda y) \quad = -2 \lambda x = \lambda (-2x) \qquad = \lambda N (x,y)$

➔ N = 1 ➔ PD homogen derajad 1

Transformasi : y = ux ➔ dy = udx + xdu

➔$(2ux + \sqrt{x^2 + 4u^2 x^2}\ ) dx – 2x (u\ dx + x\ du) = 0$

➔ $x\ dx + 2uxdx + 2\ uxdx + 2\ x^2\ du + 3\ u^2x\ dx + 3\ ux^2du = 0$

➔ $(x + 2ux + 2ux + 3u^2\ x) dx + (2x^2 + 3ux^2) du = 0$

➔ $x (1 + 4u + 3u^2) dx + x^2 (2 + 3u) du = 0$

FI : $\frac{1}{x^2(1+4u+3u^2)}$

➔ $\frac{1}{x} dx + \frac{2+3u}{1+4u+3u^2}\ du = 0$

➔ $\int \frac{dx}{x} + \int \frac{2+3u}{1+4u+3u^2}\ du = C_1$ ⟺ $\int \frac{dx}{x} + \int \frac{1}{1+4u+3u^2}\ ½\ d (1 + 4u +3u^2) = C_1$

➔ $\ln x + ½ \ln / 1+4u + 3u^2/ = \ln c_2$

➔ $2\ln x + \ln / 1+4u + 3u^2/ = \ln c_3$

➔ $\ln x^2 (1 + 4u + 3u^2) = \ln c_3$

➔ $x^2\ 9\ 1 + 4u + 3u^2) = c_3$

$u = \frac{y}{x}$ ➔ $x^2 (1 + 4\ \frac{y}{x} + 3\ \frac{y^2}{x^2}\ ) = C_3$

➔ $\underline{x^2 + 4yx + 3y^2} = C\ (C=C_3)$

D. PERSAMAAN DEFERENSIAL NON HOMOGEN

Bentuk umum

$(ax + by + c) dx + (px + qy + r) dy = 0$

1. Bila $\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \lambda$

Penyelesaian PD :

i. Transformasi : $u = px + qy + r$ ➔ $(ax + by + c) = \lambda u$
ii. Rubah PD menjadi : $\lambda u\, dx + u dy = 0$

➔ $\lambda\, dx + dy = 0$

iii. Reduksi menjadi PD variabel terpisah
iv. Selesaikan PD ; $\lambda \int dx + \int dy = c$ ➔ $x + y = 0$

Contoh 1 :

Selesaikan PD berikut : $(2x – 5y + 2) dx + (10y - 4x – 4) dy = 0$

➔ $\frac{a}{p} = \frac{2}{-4} = -\frac{1}{2}$

$\frac{b}{q} = \frac{-5}{10} = -\frac{1}{2}$ $\Bigg\}$ $= -\frac{1}{2}$

$\frac{c}{r} = \frac{2}{-4} = \frac{1}{2}$

$u = 10y – 4x – 4y$ ➔ $2x – 5y + 2 = -\frac{1}{2} u$

➔ PD : $-\frac{1}{2} u\, dx + udy = 0$ ⇔ $-\frac{1}{2} dx + dy = 0$

➔ $-\frac{1}{2} dx + dy = c_1$

$-\frac{1}{2} x + y = c_1$

➔ $x - 2y = c$ ($c = - 2 c_1$)

2. Bila $\frac{a}{p} = \frac{b}{q} \neq \frac{c}{r}$ = maka diambil $\lambda = \frac{a}{p}$

Penyelesaian PD :

i. Transformasi : $Px + qy = u$ ➔ $dy = \frac{du - pdx}{q}$
ii. $ax + by = \lambda u$
iii. bentuk PD : $(\lambda u + c) dx + (u + r) (\frac{du - pdx}{q}) = 0$
iv. reduksi PD menjadi PD variabel terpeisah dan selesaikan PD
v. ganti $u = px + qy$ untuk peroleh variabel semula

Contoh 1 :

Selesaikan PD berikut : (3x + 2y + 1) dx – (3x + 2y – 2) dy = 0

Penyelesaian :

$$\left.\begin{array}{l} \frac{a}{p} = \frac{3}{3} = 1 \\ \frac{b}{q} = \frac{2}{2} = 1 \\ \frac{c}{r} = \frac{1}{-1} = -1 \end{array}\right\} \lambda = 1$$

Transformasi : u = 3x + 2y ➔ dy = $\frac{du - 3dx}{2}$

= 1 ➔ 3x + 2y = u

➔ PD : (u + 1) dx – (u – 1) ($\frac{du - 3dx}{2}$) = 0

➔ 2 (u + 1) dx – (4du – 3udx – du + 3dx) = 0

➔ 2u + 2 + 3u - 3 dx – (u – 1) du = 0

➔ 5u – 1 dx – u – 1 du = 0

FI : $\frac{1}{5u - 1}$, maka

➔ dx - $\frac{u - 1}{5u - 1}$ du = 0

$$\frac{u - 1}{5u - 1} = \frac{A}{5} + \frac{B}{5u - 1}$$

$$= \frac{A\,(5u - 1)}{5\,(5u - 1)} + \frac{5B}{5\,(5u - 1)}$$

$$= \frac{\frac{1}{5}(A5u - A) + 5B)}{5u - 1}$$

Penyamaan koefisien :

u : 1 = 1/5 . A5 = A ====> A = 1

u^0 : -1 = -1/5 A + 1/5 . 5 . B ===> -1 = - $\frac{1}{5}$. 1 + 1 B ➔B = - $\frac{4}{5}$

➔ dx - $\frac{1}{5}$ + $\frac{-4}{(5u - 1)}$ du = 0 ⬄dx - $\frac{1}{5}$ du + $\frac{4}{5}\frac{du}{5u - 1}$ = 0

➔ dx - $\frac{1}{5}$ du + $\frac{4}{5}\frac{du}{5u - 1}$ = c_1

$x \ - \frac{1}{5} u + \frac{4}{5} , \frac{1}{5} \ln / 5u - 1) = c_1$

➔ $5x - u + \frac{4}{5} \ln / 5u - 1/ = c_2 \ (c_2 = 5c_1)$

➔ $\frac{5}{4} (5x - 4) + \ln / 5u - 10 / = c_3 \ (c_3 = \frac{5}{4} \ c_2)$

$u = 3x + 2y$ ===> $\frac{5}{4} (5x - 3x - 2y) + \ln / l5x + 10y - 1/ = c_3$

===> $\frac{5}{4} (x - y) + \ln / l5x + 10y - 1/ = c_3$

Contoh 2 :

Selesaikan PD berikut : $\frac{dy}{dx} = \frac{1-2y-4x}{1+2y+2x}$

Penyelesaian :

PD : $1 -2y - 4x \ dx - (i + 2y + 2x) \ dy = 0$

➔ $(4x - 2y - 1) \ dx + 2x + 2y + 1) \ dy = 0$

$$\left.\begin{array}{l} \frac{a}{p} = \frac{4}{2} = 2 \\ \frac{b}{q} = \frac{2}{1} = 2 \\ \frac{c}{r} = \frac{-1}{1} = -1 \end{array}\right\} \lambda = 2$$

➔ $2x + y = u$ dan $4x + 2y = 2u$

➔ $dy = du - 2 \ dx$

PD : $(2u - 1) \ dx + (u + 1) \ dy = 0$

➔ $(2u - 1) \ dx + (u + 1) \ (du - 2dx) = 0$

➔ $2u \ dx - dx + udu + du -2u \ dx - 2dx = 0$

➔ $-3 \ dx + (u + 1) \ du = 0$

➔ $-3 \ dx + (u + 1) \ du = C1$ ⇔ $-3x + \frac{1}{2} u^2 + u = c_1$

➔ $u = 2x + y$ ➔ $-3x + \frac{1}{2} 4x^2 + 4 \ xy + y^2) + 2x + y = c_1$

➔ $- 6x + 4x^2 + 4 \ y + y^2 + 4x + 2y = C_2 \ (c_2 = 2c_1)$

➔ <u>$4x^2 + y^2 + 4xy + 2y - 2x = c$</u>

3. Bila $\frac{a}{p} \neq \frac{b}{q}$

<u>Penyelesaian :</u>

i. Transformasi : u = ax + by + c == du = adx + bdy

V = px + qy + r == dv = pdx + qdy

$$\Rightarrow \quad dx = \frac{\begin{vmatrix} du & b \\ dv & q \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} \; \frac{q\,du - bdv}{aq - bp}$$

$$dy = \frac{\begin{vmatrix} a & du \\ p & dv \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} \; \frac{a\,dv - pdu}{aq - bp}$$

ii. Bentuk PD

$$u \left[\frac{q\,du - b\,dv}{aq - b\,.p}\right] + v \left[\frac{a\,dv - p\,du}{aq - b\,.p}\right] = 0$$

aq – bp ≠ 0 ===> u (q du – b dv) + v (a dv – pdu) = 0

➔ qu – pv du + (av – bu) dv =0

- - -> PD homogen

iii. Selesaikan PD homogen pada (ii)

iv. Gantikan u dan v dengan variabel semula

Contoh 1 :

Selesaikan PD berikut : (2x – 5y + 3) dx – (2x + 4y – 6) dy = 0

Penyelesaian :

a = 2   p = 2

b = -5   q = 4   ➔ $\frac{a}{p} \neq \frac{b}{q}$

c = 3   r = -6

Transformasi : u = 2x – 5y + 3 ===> du = 2dx – 5dy .... (1)

v = 2x + 4y -6 ==> dv = 2dx + 4dy .... (2)

dari (1) dan (2) ===>

$$dx = \frac{\begin{vmatrix} du & -5 \\ dv & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 2 & 4 \end{vmatrix}} = \frac{4\,du - 5dv}{8+10} \qquad \frac{4\,du + 5dv}{18}$$

$$dy = \frac{\begin{vmatrix} 2 & du \\ 2 & dv \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 2 & 4 \end{vmatrix}} = \frac{2\,dv + 2du}{18}$$

➔ PD : $u \left[\frac{4\,du + 5dv}{18}\right] - v \left[\frac{2\,dv - 2du}{18}\right] = 0$

➔ $u\,[4\,du - 5dv] - v\,(2\,dv - 2du) = 0$

➔ $(44 + 2v)\,du + (5u - 2v)\,dv = 0$

Transformasi U = zu ➔ dU = zdu + udz

➔ $[4zv + 2v]\,[zdv + vdz] + [5zv - 2v]\,dv = 0$

➔ $[4z^2 v + 2zv + 5zv - 2v]\,dv + (4zv^2 + 2v^2)\,dz$

➔ $v\,(4z^2 + 7z - 2)\,dv + v^2\,(4z + 2)\,dz$

F I : $\frac{1}{v^2(4z^2+7z-2)}$

➔ $\frac{1}{v^2(4z^2+7z-2)} \quad [\,v^2(4z^2 + 7z - 2)\,dv + v^2\left(4z^2 + 7z - 2\right) dz)\,]$

➔ $\frac{1}{v}\,dv + \boxed{\frac{4z+2}{4z^2 + 7z - 2}}\;dz = 0$

→ Dipecah menjadi pecahan parsial.

$$\frac{4z+2}{4z^2+7z-2} = \frac{4z+2}{(4z-1)\quad(z+2)} = \frac{A}{(4z-1)} + \frac{B}{(z+2)}$$

➔ $\frac{4z+2}{(4z-1)\quad(z+2)} = \frac{A\,(z+2)+(4z-1)}{(4z-1)(z+2)} = \frac{z\,(A+4B)+(2A-B)}{(4z-1)(z+2)}$

Penyamaan koefisien :

Z^1 : 4 = A + 4 B ===> A 4 – 4B .... (1)

Z^0 : 2 = 2A – B ===> 2 = 2 (4 – 4B) – B

===> 2 = 8 – 8B – B

===> -6 = -9 B ===> B = $\frac{6}{9} = \frac{2}{3}$

➔ A : $4 - 4\,(\frac{2}{3})$

$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$

➔ $\frac{4z+2}{4z^2+7z-2} = \frac{2}{3}\left(\frac{1}{4z-1}\right) + \frac{4}{3}\left(\frac{1}{3+2}\right)$

➔ $\frac{1}{v}$ dv + $\frac{2}{3}$ $\frac{dz}{(4z-1)}$ + $\frac{4}{3}$ $\frac{dz}{(z+2)}$ = 0

➔ $\int \frac{dv}{v}$ + $\frac{2}{3}$ $\int \frac{dz}{(4z-1)}$ + $\frac{4}{3}$ $\int \frac{dz}{(z+2)}$ = c_1

➔ ln v + $\frac{2}{3}$ ln / z+2/ + $\frac{4}{3}$. $\frac{1}{4}$ ln /4z-1/ = c_1

➔ 3 lnv + 2ln /z+2/ + ln /4z-1/ = c_2 (c_2 = 3 c_1)

➔ ln v^3 + ln (z+2)^2 + ln (4z-1) = c_2

➔ ln [v^3 (z+2)^2 (4z-1)] = c_2

➔ v^3 (z+2)^2 (4z-1) = c_3 (c_3 = c^{c_2})

z = $\frac{u}{v}$ ➔ v^3 ($\frac{u}{v}$ + 2)^2 (4 $\frac{u}{v}$ 1) = c_3

➔ $v^3 \left[\frac{4u-v}{v}\right] \left[\frac{u+2v}{v}\right]^2$ = c_3

➔ [4u – v] [4 + 2v] ^{2} = c_3

u = 2x – 5y + 3 dan v = 2x + 4y - 6

➔ [4 (2x - 5y + 3) – (2x + 4y – 6)] [(2x – 5y + 3) + 2 (2x + 4y – 6)]^2 = c_3

➔ [6x – 24y + 8] [6x + 3y - 9]^2 = c_3

➔ 6 [x – 6y + 3] 3^2 [2x + y – 3]^2 = c_3

➔ [x – 6y + 3] [2x + y -3]^2 = c [c = $\frac{c_3}{36}$]

➔ [x – 6y +3] [2x + y – 3]^2 = c

E. BENTUK PERSAMAAN DEFERENSIAL : y f (x, y) dx + xg (x, y) dy = 0

Penyelesaian PD

i. Transformasi : z = xy ➔ dz = xdy + ydx

➔ dy = $\frac{dz - y\,dx}{x}$ = $\frac{dz - \frac{z}{x}\,dx}{x}$

➔ $\frac{x\,dz - z\,dx}{x^2}$

ii. Reduksi PD ke bentuk variabel terpisah

iii. Gantikan z = xy untuk peroleh variabel-variabel semula

Contoh 1 :

Selesaikan PD berikut : (xy^2 + y) dx + (x^2 y – x) dy = 0

Transformasi : z = xy ===> y = $\frac{z}{x}$ ➔ dy = $\frac{x\,dz - z\,\mathrm{dx}}{x^2}$

➔ PD : $\frac{z}{x}$ (x $\frac{z}{x}$ + 1) dx + x (x $\frac{z}{x}$ - 1) dy = 0

➔ $\frac{z}{x}$ (z + 1) dx + x (z -1) dy = 0

➔ $\frac{z}{x}$ (z + 1) dx + x (z -1) $\frac{x\,dz - z\,\mathrm{dx}}{x^2}$ = 0

➔ z (z + 1) dx + (x (z -1) dz – z (z – 1) dx) = 0

➔ z ((z + 1) - (z -1) dx + x (z – 1) dz) = 0

➔ 2z dx + (z – 1) dz = 0

➔F I : $\frac{1}{\mathrm{x\,x}}$

➔ $\frac{2}{x}$ dx + $\frac{z-1}{z}$ dz = 0

➔ z $\frac{dx}{x}$ + dz - $\frac{dz}{z}$ = 0

➔ 2 $\int \frac{dx}{x}$ + $\int dz$ - $\int \frac{dz}{z}$ = c_1

➔ 2 lnx + z – lnz = c_1

➔ ln $\frac{x^2}{z}$ + z = c_1

➔ ln $\frac{x^2}{z}$ + ln c^z = c_1

➔ ln $\frac{x^2}{z}$. e^z = c_2 (c_2 = c $^{c}{}_{i}$)

➔ $\frac{x^2}{z}$. e^z = c_2

x^2 = c_2 . z . e^{-z}

z = xy ➔ x^2 = c_2 . xy . e^{-xy}

➔ <u>x = c_2 y e^{-xy}</u>

Contoh 2:

Selesaikan PD berikut : (xy^2 + y) dx + (x + x^2 + x^3 y^2) dy = 0

Penyelesaian :

PD L : $y (xy + 1) dx + (1 + xy + x^2 y^2) dy = 0$

Transformasi : $z = xy$ ➔ $y = \frac{z}{x}$ ➔ $dy = \frac{xdz-zdx}{x^2}$

➔ PD : $\frac{z}{x} (x\frac{z}{x} + 1) dx + x (1 + x\frac{z}{x} + x (1 + x \frac{z}{x^2}\ ^2) (\frac{xdz-zdx}{x^2}) = 0$

➔ $\frac{z}{x} (z + 1) dx + x + x (1 + z + z^2) (\frac{xdz-zdx}{x^2}) = 0$

➔ $z (z + 1) dx + (1 + z + z^2) (xdz - zdx) = 0$

➔ $z (z + 1) - 1 + z + z^2 \ dx + x (i + z + z^2) dz = 0$

➔ $- z^3 dx + x (1 + z + z^2) dz = 0$

➔ F I : $\frac{1}{- z^3 x}$

➔ $\frac{dx}{x} - \frac{1+2+z^3}{z^3} dz = 0$

➔ $\frac{dx}{x} - \frac{dz}{z^3} - \frac{dz}{z^2} - \frac{dz}{z} = 0$

➔ $\int \frac{dx}{x} - \int \frac{dz}{z^3} - \int \frac{dz}{z^2} - \frac{dz}{z} = c_1$

➔ $\ln x + \frac{1}{2 z^2} + \frac{1}{z} - \ln z = c_1$

➔ $\ln \frac{x}{z} + \frac{1}{2 z^2} + \frac{1}{z} = c_1$

➔ $2z^2 \ln \frac{x}{z} + 1 + 2z = c_2 z^2$ ($c_2 = 2 c_1$)

$z = xy$ ===> $2 x^2 y^2 \ln \frac{1}{z} + 1 + 2xy = c_2 x^2 y^2$

➔ <u>$1 + 2xy - x^2 y^2 \ln y = c\ x^2 y^2$</u> ($c = c_2$)

F. PERSAMAAN DEFERENSIAL EKSAK

Bentuk umum : $M (x, y) dx + N (x, y) dy = 0$

PD eksak ➔ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Penyelesaian umum PD : $f (x, y) = c_1$

Penyelesaian :

i. Perlihatkan bahwa $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

ii. Integralkan M (x, y) terhadap x dengan y tetap

$$\frac{\partial f}{\partial x} dx = M (x, y) dx$$

➔ f (x, y) = $\int^{x} M(x, y) dx + Ø (y)$

- Ø (y) = fungsi sembarang untuk y

iii. f (x, y) hasil (ii) dideferensialkan pertialkan terhadap y :

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\int^{x} M(x, y) dx \right] + \frac{\partial Ø}{\partial y}$$

iv. $\frac{\partial f}{\partial y}$ = N (x, y), maka dengan hasil (iii) diperoleh :

$$\frac{\partial Ø}{\partial y} N (x, y) - \frac{\partial}{\partial y} \left[\int^{x} M(x, y) dx \right]$$

➔ diperoleh Ø (y) = $\int \frac{\partial Ø}{\partial y} dy$

v. Ø (y) masuk ke f (x, y) dari (ii)

Contoh 1 :

Selesaikan PD berikut :

(x^2 – y) dx – x dy = 0

Penyelesaian :

M (x, y) = x^2 – y ===> $\frac{\partial M}{\partial y}$ = 0 – 1 = -1

N (x, y) = - x ====> $\frac{\partial N}{\partial x}$ = -1

(keduanya) $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➔ f (x, y) = C

$\frac{\partial f}{\partial x}$ = M (x, y) ===> f (x, y) = $\int^{x} (x^2 - y) dx$ =

= x^3 – yx + Ø (y) .... (1)

$\frac{\partial f}{\partial y}$ = N (x, y) ➔ $\frac{\partial}{\partial y}$ ($\frac{1}{3}$ x^3 - yx + Ø (y) = -x

➔ $0 - x + \frac{\partial}{\partial y}$ Ø (y) = -x

➔ $\frac{\partial}{\partial y}$ Ø (y) = 0 ➔ Ø (y) = $\int 0 \, dy$ = c_2 .... (2)

➔ f (x, y) = $\frac{1}{3}x^3 - yx + c_2$

➔ $c_1 - c_2 = \frac{1}{3}x^3 - yx$

➔ $\frac{1}{3}x^3 - yx = c$ ($xC = c_1 - c_2$)

Contoh 2

Selesaikan PD berikut : [$x^2 - y^2$] dx + 2 xy dy = 0

Penyelesaian :

M (x, y) = $x^2 + y^2$ ===> $\frac{\partial M}{\partial y} = 0 + 2y = 2y$

N (x, y) = 2 xy ====> $\frac{\partial N}{\partial x}$ = 2y

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➔ f (x, y) = c_1

$\frac{\partial f}{\partial x}$ = M (x, y) ===> f (x, y) = $\int^x (x^2 + y^2)$ x

= $\frac{1}{3}x^3 + y^2$ + Ø (y) .... (1)

$\frac{\partial f}{\partial y}$ = N (x, y)

➔ $\frac{\partial}{\partial y}$ ($\frac{1}{2}x^3 + y^2$ + Ø (y) = 2 xy

➔ 0 + 2 xy + $\frac{\partial}{\partial y}$ Ø (y) = 2 xy

➔ $\frac{\partial}{\partial y}$ Ø (y) = 0 ➔ Ø (y) = c_2 .... (2)

➔ f (x, y) = $\frac{1}{3}x^3 + yx + c_2$

➔ $c_1 - c_2 = \frac{1}{3}x^3 + y^2 x$

➔ $\frac{1}{3}x^3 + y^2 x = c \quad (c = c_1 - c_2)$

Contoh 3

Selesaikan PD berikut : $(2x + 3y + 4)\, dx + (3x + 4y + 5)\, dy = 0$

Penyelesaian :

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (2x + 3y + 4) = 0 + 3 + 0 = 3$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (3x + 4y + 5) = 3 + 0 + 0 = 3$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➔ $f (x, y) = c_1$

$\frac{\partial f}{\partial x} = M (x, y) \Longrightarrow f (x, y) = \int^{x} (2x + 3y + 4)\, dx$

$= x^2 + 3yx + 4x + \varnothing (y)$ .... (1)

$\frac{\partial f}{\partial y} = N (x, y)$ ➔ $\frac{\partial}{\partial y} (x^2 + 3yx + 4x + \varnothing (y) = 3x + 4y + 5$

➔ $0 + 3y + 0 + \frac{\partial}{\partial y} \varnothing (y) = 3x + 4y + 5$

➔ $\frac{\partial}{\partial y} \varnothing (y) = 4y + 5$

➔ $\varnothing (y) = \int (4y + 5)\, y$

$= 2y^2 + 5y + c_2$ .... (2)

➔ $f (x, y) = x^2 + 3yx + 4x + 2y^2 + 5y + c_2$

➔ $c_1 - c_2 = x^2 + 3yx + 4x + 2y^2 + 5y$

➔ $x^2 + 2y^2 + 3yx + 4x + 5y = c \quad (c = c_1 - c_2)$

G. REDUKSI KE PERSAMAAN DEFERENSIAL EKSAK

Bentuk umum : $M (x, y)\, dx + N (x, y)\, dy = 0$

Tetapi $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Dengan F I : u (x , y) , sehingga PD menjadi

$u(x, y)[M(x, y)\,dx + N(x, y)\,dy] = 0$

Jenis –jenis F I

a. Bila $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ = f (x) atau hanya fungsi x ➔ F I : $e^{\int f(x)\,dx}$
b. Bila $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ = - g (y) atau hanya fungsi y ===> F I = $e^{\int +g(y)\,dy}$
c. Bila M (x, y) dx + N (x, y) dy = 0 adalah PD homogen dan

   x M (x, y) + y N (x, y) ≠ 0 ===> F I : $\frac{1}{xM+yN}$

d. Bila PD dapat dituliskan sebagai berikut :
   y f(x, y) dx + g(x, y) dy = 0 dan f(x, y) dx ≠ g(x, y), maka
   F I : $\frac{1}{xM - yN}$
e. Persamaan $x^p y^q$ (my dx + nx dy) + $x^r y^s$ (uy dx + vx dy) = 0

   Dengan p, q, r, s m, n, u, v adalah konstanta dan mv – nu ≠ 0, maka F I : $x^{\propto} y^{\beta}$

f. F I dengan cara coba-coba :

Penyelesaian PD :

i. Tunjukkan PD tidak eksak $\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right]$
ii. Tentukan F I yang tidak cocok
iii. Untuk F I jenis a sampai d maka pD menjadi eksak dan diselesaikan menurut PD eksak
iv. Bila F I jenis e yang cocok, cari harga dan yang cocok

Contoh 1

Selesaikan PD berikut : (2y – x^3) dx + xdy = 0

Penyelesaian :

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2y - x^3) = 2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x) = 1$$

$$\Big\} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Menentukan F I : $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x}$ ==> f (x)

➔ F I = $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{lnx}$ = x

➔ PD : x (2y – x^3) dx + x . x . dx = 0

===> $(2 xy - x^4)dx + x^2 dx = 0$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (2 xy - x^4) = 2x$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (x^2) = 2x$

$\left.\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ PD eksak

$\Rightarrow f (x, y) = c_1$

$\frac{\partial f}{\partial x} = M (x, y)$ ===> $f (x, y) = \int^x (2 xy - x^4) x$

$= x^2 y - \frac{1}{5} x^5 + \emptyset (y)$ .... (1)

$\frac{\partial f}{\partial y} = N (x, y) \quad \Rightarrow \frac{\partial}{\partial y} (x^2 y - \frac{1}{5} x^5 + \emptyset (y) = x^2$

$\Rightarrow x^2 - 0 + \frac{\partial}{\partial y} \emptyset (y) = x^2$

$\Rightarrow \frac{\partial}{\partial y} \emptyset (y) = 0$ ===> $\emptyset (y) = c_2$

$\Rightarrow f (x, y) = x^2 y - \frac{1}{5} x^5 + c_2$

$\Rightarrow c_1 - c_2 = x^2 y - \frac{1}{5} x^5$

$\Rightarrow x^2 y - \frac{1}{5} x^5 = c_2$

Contoh 2 :

Selesaikan PD berikut : $(3 x^2 y^2) dx + 4 x^3 y - 12 dy = 0$

Penyelesaian :

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (3 x^2 y^2) = 6 x^2 y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (4 x^3 y - 12) = 12 x^2 y$

Menentukan F I : $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6 x^2 y \; 12 x^2 y}{3 x^2 y^2} = - \frac{2}{y} = 9 (y)$

➔ F I = $e^{\int 9\,(y)dx}$ = $e^{\int \frac{1}{2}\,dy}$= $e^{21\,lny}$ = e^{lny^2} = y^2

➔ PD : $y^2 (3 x^2 y^2) dx + y^2 (4 x^3 y - 12) dy = 0$

➔ $3 x^2 y^4 dx + (4 x^3 y^3 - 12 y^2) dy = 0$

➔ $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\ 3 x^2 y^4 = 12 x^2 y^4 = 12 x^2 y^3$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\ (4 x^3 y^3 - 12 y^2) = 12 x^2 y^3 - 0 = 12 x^2 y^3$

} $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➔ f (x, y) = c

$\frac{\partial f}{\partial x}$ = M (x, y) ===> f (x, y) = $\int^x \quad 3 x^2 y^4\ x$

= $x^3 y^4 + \emptyset\,(y)$ .... (1)

$\frac{\partial f}{\partial y}$ = N (x, y) ➔ $\frac{\partial}{\partial y}\ x^3 y^4 + \emptyset\,(y) = 4 x^3 y^3 - 12 y^2$

➔ $4 x^3 y^3 + \frac{\partial}{\partial y}\ \emptyset\,(y) = 4 x^3 y^3 - 12 y^2$

➔ $\frac{\partial}{\partial y}\ \emptyset\,(y) = - 12 y^2$

➔ $\emptyset\,(y) = \int^y \quad - 12 y^2\ y$

$= - 4 y^3 + c_2$

➔ $f (x, y) = x^3 y^4 - 4 y^3 + c_2$

➔ $c_1 - c_2 = x^3 y^4 - 4 y^3$

➔ $x^3 y^4 - 4 y^3 = c$ $\qquad C = c_1 - c_2$

Contoh 3:

Selesaikan PD berikut : $y (x + y) dx - x^2 dy = 0$

Penyelesaian :

PD : $(xy + y^2) dx - x^2 dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\ (xy + y^2) = x + 2y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x^2) = -2x$

F I :

M (x, y) = xy + y^2 ➔ M (λx, λy) = λx, λy + $\lambda^2 y^2 = \lambda^2 (xy + y^2)$

N (x, y) = - x^2 ➔ N (λx, λy) = $\lambda^2 y^2$ = $\lambda^2 (-x^2)$

➔ PD homogen derajad 2

x (M (x, y) + y N (x, y) = x (xy + y^2) + y (-x^2) = $y^2 x$

➔ F I : $\frac{1}{y^2 x}$

➔ PD : $\frac{1}{y^2 x}(xy + y^2)\, dx + \frac{1}{y^2 x}(-x^2)\, dy = 0$

➔ $\left[\frac{1}{y} + \frac{1}{x}\right] dx - \frac{x}{y^2}\, dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y} + \frac{1}{x}\right) = -\frac{1}{y^2} + 0 = -\frac{1}{y^2}$

$\frac{\partial N}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{y^2}\right) = -\frac{1}{y^2}$

} $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➔ f (x, y) = c_1

$\frac{\partial f}{\partial x}$ = M (x, y) ===> f (x, y) = $\int^x \left[\frac{1}{y} + \frac{1}{x}\right] x$

$= \frac{x}{y} + \ln /x/ + \emptyset (y)$ .... (1)

$\frac{\partial f}{\partial y}$ = N (x, y) ➔ $\frac{\partial}{\partial y}\left(\frac{x}{y} + \ln /x/ + \emptyset (y)\right) = -\frac{x}{y^2}$

➔ $-\frac{x}{y^2} + 0 + \frac{\partial}{\partial y} \emptyset (y) = -\frac{x}{y^2}$

➔ $\frac{\partial}{\partial y} \emptyset (y) = 0$ ➔ $\emptyset (y) = c_2$

➔ f (x, y) = $\frac{x}{y} + \ln /x/ + c_2$

➔ $c_1 - c_2 = \frac{x}{y} + \ln /x/$ ⬌ $\frac{x}{y} + \ln /x/ = C$

Contoh 4 :

Selesaikan PD berikut : (x^2 y^2 + 2y) dx + (2x – 2x^3 y^2) dy = 0

Penyelesaian :

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}$ (x^2 y^3 + 2y) = 3 x^2 y^2 + 2

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial x}$ = $\frac{\partial}{\partial x}$ (2x – 2x^3 y^2) = 2- 6 x^2 y^2

M (x, y) = x^2 y^3 + 2y ➔ M (λx, λy) = λ^2 x^2 , λ^3 y^3 + 2 λ ≠ λ^n M(x , y)

➔ PD non homogen

PD : y (x^2 y^2 + 2y) dx + x (2 – 2 x^2 y^2) dy = 0

➔ F I : $\frac{1}{xM - yN} = \frac{1}{x^3y^3 - 2xy - 2xy + 2x^3y^3} = \frac{1}{3x^3\, y^3}$

➔ PD : $\frac{1}{3x^3\, y^3}$ y^3 (x^2 y^3 + 2y) dx + $\frac{1}{3x^3\, y^3}$ (2x – 2x^3 y^2) dy = 0

➔ $\left[\frac{1}{3x} + \frac{2}{3x^3\, y^2}\right]$ dx - $\left[\frac{2}{3x^2\, y^3} + \frac{2}{3y}\right]$ dy = 0

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}$ $\left(\frac{1}{3x} + \frac{2}{3x^3\, y^2}\right)$ = - $\frac{4}{3x^3\, y^3}$ = - $\frac{4}{3x^3\, y^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial y} = \frac{\partial}{\partial y}$ $\left(\frac{2}{3x^2\, y^3} - \frac{2}{3y} \right)$ = - $\frac{4}{3x^3\, y^2}$

➔ f (x, y) = c

$\frac{\partial f}{\partial x}$ = M (x, y) ===> f (x, y) = $\int^x \left[\frac{1}{3x} + \frac{2}{3x^3\, y^2}\right]$ y

= $\frac{1}{3}$ + ln x - $\frac{1}{3x^2\, y^2}$ + Ø (y) .... (1)

$\frac{\partial f}{\partial y}$ = N (x, y) ➔ $\frac{\partial}{\partial y}$ [$\frac{1}{3}$ + ln x - $\frac{1}{3x^2\, y^2}$ + Ø (y)] = $\frac{2}{3x^2\, y^3}$ - $\frac{2}{3y}$

➔ 0 + $\frac{2}{3x^2\, y^3}$ + $\frac{\partial}{\partial y}$ Ø (y) = $\frac{2}{3x^2\, y^3}$ - $\frac{2}{3y}$

➔ $\frac{\partial}{\partial y}$ Ø (y) = - $\frac{2}{3y}$

➔ $\emptyset$ (y) $= \int^{y} - \frac{2}{3y}$ y

$= - \frac{2}{3} \ln y + c_1$

➔ $f(x, y) = \frac{1}{3} \ln x - \frac{1}{3x^2 y^2} - \frac{2}{3} \ln y + c_1$

➔ $c_1 - c_2 = \frac{1}{3} \ln x - \frac{2}{3} \ln y - \frac{1}{3x^2 y^2}$

$c_3 = \ln x - 2 \ln y - \frac{1}{x^2 y^2}$ [$c_3 = 3 (c_1 - c_2)$]

$C_1 = \ln \frac{x}{y^2} - \ln e \ \frac{1}{x^2 y^2} = \ln \frac{x}{y^2} + \ln e^{x^2 y^2}$

➔ $c_3 = \left[\frac{x}{y^2} e^{x^2 y^2}\right]$

➔ $\frac{x}{y^2} e^{x^2 y^2} = C$ [$C = e^{c_3}$]

➔ $x = \underline{C y^2 e^{-x^2 y^2}}$

Contoh 5 :

Selesaikan PD berikut : $8 y\, dx + 8 x dy + x^2 y^2 (4 y dx + 5 x\, dy) = 0$

Penyelesaian :

F I : $x^{\propto} y^{\beta}$

➔ PD : $x^{\propto} y^{\beta}$ [$8 y\, dx + 8 x dy + x^2 y^2 (4 y dx + 5 x\, dy)$] = 0

=> $\underbrace{8 x^{\propto} y^{\beta+1} dy + 8 x^{\propto+1} y^{\beta} dy}_{I} + \underbrace{4^{\propto+2} y^{\beta+4} dx + 5x^{\propto+3} y^{\beta+3} dy}_{II} = 0$

I : $d [x^{\propto+1} y^{\beta+1}] = (\propto+1)^{\propto} x\ y^{\beta+1} dx + (\beta+1) x^{\propto+1} y^{\beta} dy$

➔ $\frac{\propto+1}{8} = \frac{\beta+1}{8}$ ➔ $\propto = \beta$

II : $d (x^{\propto+3} y^{\beta+4}] = (\propto+3) x^{\propto+2} y^{\beta+4} dx + (\beta+4) x^{\propto+3} y^{\beta+3} dy$

➔ $\frac{\propto+3}{4} = \frac{\beta+4}{5}$ ⟺ $5\propto + 15 - 4 \beta + 16$ ⟺ $5\propto = 4\beta+1$

➔ (1) ke (2) : $5\propto = 4 \propto + 1$ ➔ $\propto = 1$ dan $\beta = 1$

➔ F I : x y

➔ PD : $8 xy^2 dx^2 + 8 x^2 y\, dy + 4 x^3 y^5 dx + 5 x^4 y^4 dy = 0$

=====> $[8 xy^2 + 4 x^3 y^5] dx + [8 x^2 y + 5 x^4 y^4] dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}$ (8 xy^2 + 4 $x^3 y^5$)= 16 xy + 20 $x^3 y^4$

$\frac{\partial N}{\partial y} = \frac{\partial}{\partial y}$ [8 x^2 y + 5 $x^4 y^4$] = 16 xy + 20 $x^3 y^4$

} $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

➔ f (x, y) = c_1

$\frac{\partial f}{\partial x}$ = M ===> f (x, y) = $\int^x$ [8 xy^2 + 4 $x^3 y^5$] ∂x

= 4 $x^2 y^2$ + $x^4 y^5$ Ø (y) .... (1)

$\frac{\partial f}{\partial y}$ = N ===> ➔ $\frac{\partial}{\partial y}$ [4 $x^2 y^2$ + $x^4 y^5$ Ø (y)] = 8 $x^2 y^2$ + 5 $x^4 y^4$

➔ 8 x^2 + 5 $x^4 y^4$ + $\frac{\partial}{\partial y}$ Ø (y) = 8 $x^2 y^2$ + 5 $x^4 y^4$

➔ $\frac{\partial}{\partial y}$ Ø (y) = 0 ===> Ø (y) = c_2

➔ f (x, y) = 4 $x^2 y^2$ + $x^4 y^5$ + c_2

➔ c_1 - c_2 = 4 $x^2 y^2$ + $x^4 y^5$

➔ <u>4 $x^2 y^2$ + $x^4 y^5$ = 6</u> (c = c_1 - c_2)

H. PERSAMAAN DEFERENSIAL LINIER DERAJAD SATU

Bentuk umum : $\frac{dy}{dx}$ + y (P (x)) = Q (x)

F I : $e^{\int P(x)dx}$

Penyelesaian umum :

y $e^{\int P(x)dx}$ = $\int Q(x)\ e^{\int P(x)dx}$ dx+ C

Penyelesaian :

i. Tentukan F I
ii. Integrasikan ruas kanan penyesuaian umum

Contoh 1 :

Selesaikan PD berikut : $\frac{dy}{dx}$ + y = 2 + 2x

Penyelesaian :

P (x) = 1

Q (x) = 2 + 2x

$\left.\begin{matrix} P(x) = 1 \\ Q(x) = 2+2x \end{matrix}\right\}$ F I = $e^{\int P(x)dx}$ = $e^{\int dx}$ = c^x

➔ y e^x = $\int (2 + 2x)\ e^x$ dx

= $\int$ 2 e^x + 2 (xe^x - $\int$ e^x dx) + c

= 2 e^x + 2 x e^x – 2 e^x + c

= 2 x e^x + c

y = 2x + c e^{-x}

Contoh 2 :

x $\frac{dy}{dx}$ = y (1 – x tgx) + x^2 cos x

Penyelesaian :

PD : x $\frac{dy}{dx}$ - y [1 – x tgx] + x^2 cos x

➔ $\frac{dy}{dx}$ + y [tgx - $\frac{1}{x}$] + x cos x

➔ P (x) = tgx - $\frac{1}{x}$

Q (x) = x cosx

FI = $e^{\int P(x)dx}$

= $e^{\int \left(tg - \frac{1}{x}\ \right)dx}$

= $e^{-\ln x \cos x - \ln x}$

= $e^{\ln x \cos x}$

= $e^{\ln}$ $\frac{1}{x\ cosx}$ = $\frac{1}{x\ cosx}$

➔ y $\frac{1}{x\ cosx}$ = $\int$ Q (x) . $\frac{1}{x\ cosx}$ dx

= $\int x \cos . \frac{1}{x\ cosx}\ \ dx = \ \int dx = x + c$

➔ y = x^2 cosx + c x cosx

Contoh 3 :

Selesaikan PD berikut : $\frac{dy}{dx}$ = 2 $\frac{y}{x}$ x^2 e^x

Penyelesaian :

PD : $\frac{dy}{dx} - y\ \frac{2}{x} = x^2\ e^x$

➔ P (x) = - $\frac{2}{x}$

FI = $e^{\int P\ (x) dx} = e^{\int -\frac{2}{x} dx} = e^{-2\ lnx} = e^{-\ lnx^2}$

Q (x) = $x^2\ e^x$

= $e^{\ln \frac{1}{x^2}} = \frac{1}{x^2}$

➔ y . $\frac{1}{x^2}$ $= \int Q\ (x).\ \frac{1}{x^2}\ dx$

= $\int x^2.\ e^x.\ \frac{1}{x^2}\ dx = \int e^x\ dx$

= e^x + c

➔ t = x^2 e^x + c x^{-2}

I. PERSAMAAN DEFERENSIAL BERNOULLI

Bentuk PD : $\frac{dy}{dx} + y\ P\ (x) = y^n\ Q\ (x), dan\ n\ \neq 0$

Transformasi : z = y^{-n+1} ➔ $\frac{dz}{dx} = (\ 1 - n)\ y^{-n}\ \frac{dy}{dx}$

➔ $\frac{dy}{dx} = \frac{1}{(1-n)}\ y^n\ \frac{dz}{dx}$

➔ PD : $\frac{1}{(1-n)}\ y^n\ \frac{dz}{dx} + z\ y^n\ P(x) = \ y^n\ \ Q(x)$

➔ $\frac{dz}{dx} +\ (1-n)z\ P(x) = Q(x)\ (1-n)$

Penyelesaian umum : $ze^{\int (1-n)\ P(x) dx = \int Q(x)\ (1-n)\ e^{\int (1-n)\ P(x) dx}}$

Penyelesaian :

i. Transformasi : z = y^{1-n} ➔ $\frac{dy}{dx} = \frac{1}{(1-n)}\ y^n\ \frac{dz}{dx}$
ii. Selesaikan PD
iii. Transformasikan z dg y

Contoh 1 :

$\frac{dy}{dx} - y = xy^2$

Penyelesaian :

P (x) = 1

Q (x) = x dan n = 2

Transformasi : z = y^{1-n} = y^{1-2} = y^{-1} = $\frac{1}{y}$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{1-2} \; y^{-2} \; \frac{dy}{dx} = -y^{-2} \; \frac{dy}{dx}$$

$\Rightarrow$ PD : $-y^2 \; \frac{dz}{dx} - y = xy^2$

$\Rightarrow \frac{dz}{dx} + \frac{1}{y} = -x \Leftrightarrow \frac{dz}{dx} + z = -x$

$\Rightarrow$ P (x) = 1
Q (x) = - x

FI = $e^{\int P(x)dx} = e^{dx} = e^x$

$\Rightarrow$ z $e^x = \int Q\,(x)e^x \;\; dx = \int -x\, e^x\, dx$

= $-x\, e^x + e^x + c$

= e^x (1 - x) + c

$\Rightarrow$ z = 1 – x + c e^{-x}

z = $\frac{1}{y} \Rightarrow \frac{1}{y}$ = 1 – x + c e^{-x}

= $\frac{1}{y}$ = 1– x + c e^{-x}

Contoh 2 :

Selesaikan PD berikut :

X dy +y dx = x^3 y^6 dx

Penyelesaian :

PD : $x \; \frac{dy}{dx} + y = x^3 y^6$

n =: 6 ===> transformasi : z = y^{1-n} = y^{1-c} = y^{-5} = $\frac{1}{y^5}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-6} \; y^6 \; \frac{dz}{dx}$$

$$= -\frac{1}{5} \; y^6 \frac{dz}{dx}$$

➔ PD : $-\frac{1}{5}\, y^6 \frac{dz}{dx} + \frac{y}{x} = x^2 y^6$

➔ $\frac{dz}{dx} - \frac{5}{x}\, y^{-5} = -5\, x^2$ ⟺ $\frac{dz}{dx} - \frac{5}{x}\, z = -\, x^2$

➔ P (x) = $-\frac{5}{x}$

FI : $e^{\int P\,(x) dx}$ = $e^{-5\,\int \frac{dx}{x}}$ = $e^{-5\, lnx}$ = $e^{\ln \frac{1}{x^5}}$

Q (x) = - 5 x^2

➔ z x^{-5} = - 5 $\int x^2\, x^{-5}\, dx = -5\, \int x^{-3}\, dx$

= $\frac{5}{2}\, x^{-2} + c$

➔ z x^{-5} = $\frac{5}{2}\, x^{-2} + c$

➔ z = $\frac{5}{2}\, x^3 + c\, x^5$

z = y^{-5} ➔ y^{-5} = $\frac{5}{2}\, x^3 + c\, x^5$

Contoh 3 Selesaikan PD berikut:

$x\, \frac{dy}{dx} + y = xy^2 \ln x$

Penyelesaian :

PD : $\frac{dy}{dx} + \frac{y}{x} = y^2 \ln x$

n = 2 ===> transformasi z = y^{-2} = y^{-1} = $\frac{1}{y}$

===> $\frac{dy}{dx} = \frac{1}{1-2}\, y^2\, \frac{dz}{dx} = -y^2\, \frac{dz}{dx}$

➔ PD : - $y^{-2} \frac{dz}{dx} + \frac{y}{x} = y^2 \ln x$

➔ $\frac{dz}{dx} - \frac{1}{x}$ y^{-1} = - lnx

➔ $\frac{dz}{dx} - \frac{1}{x}$ z = - lnx

P (x) = $-\frac{1}{x}$

FI = $e^{-\int \frac{1}{x} dx}$ = e^{-lnx} = $e^{\ln \frac{1}{x}}$ = $\frac{1}{x}$

Q (x) = - lnx

➔ z . $\frac{1}{x} = -\int \ln x\, .\, \frac{1}{x}\, dx = -\int \ln x\, d\, lnx = -\frac{1}{2}\, ln^2\, x + c$

➔ $z = cx - \frac{1}{2}\, ln^2 x$

$z = \frac{1}{y}$ ➔ $\frac{1}{y} = cx - \frac{1}{2}\, x \ln x$

J. Nilai Awal (Initial Value Problem)

Contoh 1 :

Selesaikan PD berikut :

$$x\,\frac{dy}{dx} + y = 0, y\,(1) = 1$$

Penyelesaian :

PD : x dy + y dx = 0

➔ $\frac{dy}{y} + \frac{dy}{x} = 0$

➔ $\int \frac{dy}{y} + \int \frac{dx}{x} = c_1$

➔ $\ln y + \ln x = c_1$

➔ $\ln yx = \ln c_2$ ($c_1 = \ln c_2$)

$yx = c_2$ ====> $c_2 = 1$

➔ $y\,(1) = 1$ ===> $= c_2 \,.\, 1^{-1} = \frac{1}{x}$

Contoh 2 :

Selesaikan PD berikut :

$$\frac{dy}{dx} = \frac{y+x}{y-x}\,.\,y\,(0) = 2$$

Penyelesaian :

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial u}\,(y + x) = 1$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\,(-y + x) = 1$

} PD eksak

➔ $f\,(x,y) = c_1$

$\frac{\partial f}{\partial x} = M\,(x,y)$ ➔ $f\,(x,y) = \int^x (x+x)\,dx = \frac{1}{2}\,x^2 + yx + Ø\,(y)$ 1

== $\frac{\partial f}{\partial y} = N\,(x,y)$ ➔ $\frac{\partial}{\partial y}\,(\frac{1}{2}\,x^2 + yx + Ø\,(y) = x - y$

➔ $0 + x + \frac{\partial}{\partial y} \emptyset(y) = x - y$

➔ $\frac{\partial}{\partial y} \emptyset(y) = -y$

➔ $\emptyset(y) = \int^{y} -y \, \partial y = -\frac{1}{2} y^2 + c_2$

➔ $f(x,y) = \frac{1}{2} x^2 + xy - \frac{1}{2} y^2 + c_2$

$c_2 - c_2 = \frac{1}{2} x^2 + xy - \frac{1}{2} y^2$

$C = x^2 + 2xy - y^2$ $\quad (c = 2 (c_2 - c_2))$

y (0) = 2 ===> $c = 0 + 0 - 2^2$ ➔ $c = -4$

===> $-4 = x^2 - y^2 + 2xy + 4 = 0$

Contoh 3 :

Selesaikan PD berikut :

$\frac{dy}{dx} + y = e^{x}$, y (1) = 0

Penyelesaian :

$P(x) = 1$

$Q(x) = e^x$

$FI = e^{\int P(x)dx} = e^{\int dx} = e^x$

$y\, e^x = \int e^x \, . \, e^x \, dx = \int e^x \, de^x = \frac{1}{2} e^{2x} + c$

➔ $y = \frac{1}{2} e^x + ce^{-x}$

y (1) = 0 ===> $0 = \frac{1}{2} e + c\, e^{-1}$ ➔ $c = \frac{\frac{1}{2}e}{e^{-1}} = -\frac{1}{2} e^2$

➔ $y = \frac{1}{2} e^x - \frac{1}{2} e^{2-x}$