15. $\alpha + \frac{1}{\alpha} \in \mathbb{Z}$ Buktikan

$$\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}, \forall\, n \in \mathbb{N}$$

Solusi:

a) untuk n=1

$\alpha^1 + \frac{1}{\alpha^1} = \alpha + \frac{1}{\alpha} \in \mathbb{Z}$ (Benar)

b) Diasumsikan BENAR Untuk n=k

$$\alpha^k + \frac{1}{\alpha^k} \in \mathbb{Z}$$

c) Akan dibuktikan untuk n=k+1 BENAR

$$\alpha^{k+1} + \frac{1}{\alpha^{k+1}} = \alpha^k \cdot \alpha + \frac{1}{\alpha^k \cdot \alpha} = \left(\alpha + \frac{1}{\alpha}\right)\left(\alpha^k + \frac{1}{\alpha^k}\right) - \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right)$$

Untuk $\left(\alpha + \frac{1}{\alpha}\right) \in \mathbb{Z}$, BENAR (Berdasarkan n=1)

Untuk $\left(\alpha^k + \frac{1}{\alpha^k}\right) \in \mathbb{Z}$, BENAR (Berdasarkan n=k)

Untuk $\left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right) \in \mathbb{Z}$, BENAR (untuk $\forall\, n \in \mathbb{N}, n > 0$)

Bukti:

Kita mempunyai $\alpha^0 + \frac{1}{\alpha^0} = 2 \in \mathbb{Z}$, dengan asumsi bahwa $\alpha^k + \frac{1}{\alpha^k} \in \mathbb{Z}$, anggap untuk $n \in \mathbb{N}$

Akibatnya

$$\alpha^{k-1} + \frac{1}{\alpha^{k-1}} \in \mathbb{Z}$$

Sehingga:

$$\alpha^{k+1} + \frac{1}{\alpha^{k+1}} = \left(\alpha + \frac{1}{\alpha}\right)\left(\alpha^k + \frac{1}{\alpha^k}\right) - \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right)$$

Berdasarkan sifat tertutup bilangan bulat,

$$\mathbb{Z} \cdot \mathbb{Z} - \mathbb{Z} = \mathbb{Z}^2 - \mathbb{Z} \in \mathbb{Z}$$

Jadi terbukti $\forall \alpha \in \mathbb{R}, \ni\ \alpha + \frac{1}{\alpha} \in \mathbb{Z}$ berlaku $\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}, \forall\, n \in \mathbb{N}$

16. Buktikan $1 < \frac{1}{n+1} + \cdots + \frac{1}{3n+1} < 2$

Solusi:

a) Untuk $n = 1$

$$f(n) = \frac{1}{n+1} + \cdots + \frac{1}{3n+1}$$

$$f(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$jadi\ 1 < f(n) < 2$ **(BENAR)**

b) Diasumsikan **BENAR** Untuk $n = k$

$$f(n) = \frac{1}{n+1} + \cdots + \frac{1}{3n+1}$$

$$f(k) = \frac{1}{k+1} + \cdots + \frac{1}{3k+1}$$

$$\mathbf{1 < f(k) < 2}$$

c) Akan dibuktikan untuk $\boldsymbol{n = k + 1}$ juga **BENAR** $\boldsymbol{f(k) < f(k+1)}$

$$f(k+1) = \frac{1}{k+2} + \cdots + \frac{1}{3k+4}$$

$$= -\frac{1}{k+1} + \frac{1}{k+1} + \cdots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}$$

$$= f(k) - \frac{1}{k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}$$

$$= f(k) - \frac{2}{3k+3} + \frac{6k+6}{(3k+2)(3k+4)}$$

Perhatikan

$$\frac{6k+6}{(3k+2)(3k+4)} > \frac{2}{3k+3}$$

Maka:

$$\boldsymbol{f(k) < f(k+1)}$$

Karena $\boldsymbol{1 < f(k) < f(k+1)}$ maka benar jika $\boldsymbol{1 < f(n) < 2}$

17. $\forall n \in \mathbb{N}$, terdapat $f(n) = g(n)$ dengan

$$f(n) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \cdots\right) + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$g(n) = \frac{1}{n+1} + \cdots + \frac{1}{2n}$$

Adb. $f(n) = g(n)$

a) Untuk $n = 1$

$$f(1) = \left(\frac{1}{2.1-1} - \frac{1}{2.1}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$g(1) = \frac{1}{2n} = \frac{1}{2.1} = \frac{1}{2}$$

$f(1) = g(1)$

BENAR untuk $n = 1$

b) Diasumsikan BENAR untuk $n = k$

$$f(k) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \cdots\right) + \left(\frac{1}{2k-1} - \frac{1}{2k}\right)$$

$$g(k) = \frac{1}{k+1} + \cdots + \frac{1}{2k}$$

BENAR untuk $f(k) = g(k)$

c) Akan dibuktikan untuk $n = k + 1$, karena $f(k) = g(k)$ maka

$$\boldsymbol{f(k+1) - f(k) = g(k+1) - g(k)}$$

$$f(k+1) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \cdots\right) + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$$

$$f(k+1) = f(k) + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$$

$$\boldsymbol{f(k+1) - f(k) = \frac{1}{2k+1} - \frac{1}{2k+2}}$$

$$g(k+1) = \frac{1}{k+2} + \cdots + \frac{1}{2k+2}$$

$$g(k+1) = -\frac{1}{k+1} + \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$g(k+1) = g(k) - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$g(k+1) - g(k) = \frac{1}{2k+1} + \frac{1}{k+1}\left(-1 + \frac{1}{2}\right)$$

$$\boldsymbol{g(k+1) - g(k) = \frac{1}{2k+1} - \frac{1}{2k+2}}$$

Terbukti

$$\boldsymbol{f(k+1) - f(k) = g(k+1) - g(k)}$$

Jadi

$\forall n \in \mathbb{N}$, terdapat $f(n) = g(n)$ dengan

$$f(n) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \cdots\right) + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$g(n) = \frac{1}{n+1} + \cdots + \frac{1}{2n}$$

18. Buktikan $(n+1)(n+2)\dots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots (2n-1), \forall n \in \mathbb{N}$

Solusi:

$(n+1)(n+2)\dots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots (2n-1)$

$(n+1)(n+2)\dots 2n = 2^n \cdot (2n-1)!$

a) Untuk $n = 1$

$2 \cdot 1 = 2^1 \cdot (2 \cdot 1 - 1)$

$2 = 2$

BENAR

b) Diasumsikan BENAR untuk $n = k$

$(k+1)(k+2)\dots 2k = 2^k \cdot (2k-1)!$

c) Akan dibuktikan BENAR untuk $n = k+1$

$((k+1)+1)((k+1)+2)\dots 2(k+1) = 2^{(k+1)} \cdot (2(k+1)-1)!$

$(k+2)(k+3)\dots(2k+2) = 2^{(k+1)} \cdot (2k+1)!$

$(k+1)(k+2)(k+3)\dots(2k)\dfrac{(2k+1)(2k+2)}{(k+1)} = 2^{(k+1)} \cdot (2k+1)!$

$2^k \cdot (2k-1)!\dfrac{(2k+1)(2k+2)}{(k+1)} = 2^{(k+1)} \cdot (2k+1)!$

$2^k \cdot (2k-1)!\dfrac{2(2k+1)(k+1)}{(k+1)} = 2^{(k+1)} \cdot (2k+1)!$

$2^{k+1} \cdot (2k-1)!(2k+1) = 2^{(k+1)} \cdot (2k+1)!$

$2^{(k+1)} \cdot (2k+1)! = 2^{(k+1)} \cdot (2k+1)!$

Terbukti!

BENAR untuk $n = k+1$

Terbukti $(n+1)(n+2)\dots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots (2n-1), \forall n \in \mathbb{N}$