

15. $\forall \alpha \in \mathbb{R}, \exists \alpha + \frac{1}{\alpha} \in \mathbb{Z}$ Buktikan

$$\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}, \forall n \in \mathbb{N}$$

Solusi:

a) untuk $n=1$ adalah BENAR

$$\alpha^1 + \frac{1}{\alpha^1} = \alpha + \frac{1}{\alpha} \in \mathbb{Z}, \forall n \in \mathbb{N}$$

b) Diasumsikan BENAR Untuk $n=k$

$$\alpha^k + \frac{1}{\alpha^k} \in \mathbb{Z}, \forall n \in \mathbb{N}$$

c) Akan dibuktikan untuk $n=k+1$ BENAR

$$\alpha^{k+1} + \frac{1}{\alpha^{k+1}} = \alpha^k \cdot \alpha + \frac{1}{\alpha^k \cdot \alpha} = \left(\alpha + \frac{1}{\alpha}\right) \left(\alpha^k + \frac{1}{\alpha^k}\right) - \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right)$$

Untuk $\left(\alpha + \frac{1}{\alpha}\right) \in \mathbb{Z}$, BENAR (Berdasarkan $n=1$)

Untuk $\left(\alpha^k + \frac{1}{\alpha^k}\right) \in \mathbb{Z}$, BENAR (Berdasarkan $n=k$)

Untuk $\left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right) \in \mathbb{Z}$, BENAR

Bukti:

Kita mempunyai $\alpha^0 + \frac{1}{\alpha^0} = 2 \in \mathbb{Z}$, dengan asumsi bahwa $\alpha^k + \frac{1}{\alpha^k} \in \mathbb{Z}$, anggap untuk $n \in \mathbb{N}$

Akibatnya

$$\alpha^{k-1} + \frac{1}{\alpha^{k-1}} \in \mathbb{Z}$$

Sehingga:

$$P(k+1) = \left(\alpha + \frac{1}{\alpha}\right) \left(\alpha^k + \frac{1}{\alpha^k}\right) - \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right)$$

Berdasarkan teorema,

$$\mathbb{Z} \cdot \mathbb{Z} - \mathbb{Z} = \mathbb{Z}^2 - \mathbb{Z} \in \mathbb{Z}$$

Maka untuk $n = k + 1$, $\alpha^{k+1} + \frac{1}{\alpha^{k+1}} \in \mathbb{Z}$

Jadi terbukti $\forall \alpha \in \mathbb{R}, \exists \alpha + \frac{1}{\alpha} \in \mathbb{Z}$ berlaku $\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}, \forall n \in \mathbb{N}$

16. Buktikan $1 < \frac{1}{n+1} + \dots + \frac{1}{3n+1} < 2$

Solusi:

a) Untuk $n = 1$ adalah **BENAR**

$$f(n) = \frac{1}{n+1} + \dots + \frac{1}{3n+1}$$

$$f(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

jadi $1 < f(n) < 2$ (**BENAR**)

b) Diasumsikan **BENAR** Untuk $n = k$

$$f(n) = \frac{1}{n+1} + \dots + \frac{1}{3n+1}$$

$$f(k) = \frac{1}{k+1} + \dots + \frac{1}{3k+1}$$

$$1 < f(k) < 2$$

c) Akan dibuktikan untuk $n = k + 1$ juga **BENAR** $f(k) < f(k + 1)$

$$\begin{aligned} f(k+1) &= \frac{1}{k+2} + \dots + \frac{1}{3k+4} \\ &= \frac{1}{k+1} + \frac{1}{k+1} + \dots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} \\ &= f(k) - \frac{1}{k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} \\ &= f(k) - \frac{2}{3k+3} + \frac{6k+6}{(3k+2)(3k+4)} \end{aligned}$$

Perhatikan

$$\frac{6k+6}{(3k+2)(3k+4)} > \frac{2}{3k+3}$$

Maka:

$$f(k) < f(k+1)$$

Karena $1 < f(k) < f(k+1)$ maka benar jika $1 < f(n) < 2$

17. $\forall n \in \mathbb{N}$, terdapat $f(n) = g(n)$ dengan

$$f(n) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \dots\right) + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$g(n) = \frac{1}{n+1} + \dots + \frac{1}{2n}$$

Adb. $f(n) = g(n)$

a) Untuk $n = 1$

$$f(1) = \left(\frac{1}{2 \cdot 1 - 1} - \frac{1}{2 \cdot 1}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$g(1) = \frac{1}{2n} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

$$f(1) = g(1)$$

BENAR untuk $n = 1$

b) Diasumsikan BENAR untuk $n = k$

$$f(k) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \dots\right) + \left(\frac{1}{2k-1} - \frac{1}{2k}\right)$$

$$g(k) = \frac{1}{k+1} + \dots + \frac{1}{2k}$$

BENAR untuk $f(k) = g(k)$

c) Akan dibuktikan untuk $n = k + 1$, karena $f(k) = g(k)$ maka

$$f(k+1) - f(k) = g(k+1) - g(k)$$

$$f(k+1) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \dots\right) + \left(\frac{1}{2k-1} - \frac{1}{2k+2}\right)$$

$$f(k+1) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \dots\right) + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$$

$$f(k+1) = f(k) + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$$

$$f(k+1) - f(k) = \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$$

$$g(k+1) = \frac{1}{k+2} + \dots + \frac{1}{2k+2}$$

$$g(k+1) = -\frac{1}{k+1} + \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$g(k+1) = g(k) - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$g(k+1) - g(k) = -\frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$\text{Adb. } f(k+1) - f(k) = g(k+1) - g(k)$$

$$f(k+1) - f(k) = \frac{1}{2k+2} - \frac{1}{2k+1} + \frac{1}{2k+3} - \frac{1}{2k+2}$$

$$\begin{aligned} g(k+1) - g(k) &= -\frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2} \\ &= \frac{1}{2(k+1)(2k+1)} \\ &= \frac{1}{2k+1} - \frac{1}{2k+2} = f(k+1) - f(k) \end{aligned}$$

Terbukti

$$f(k+1) - f(k) = g(k+1) - g(k)$$

Jadi

$\forall n \in \mathbb{N}$, terdapat $f(n) = g(n)$ dengan

$$f(n) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \dots\right) + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$g(n) = \frac{1}{n+1} + \dots + \frac{1}{2n}$$

18. Buktikan $(n + 1)(n + 2) \dots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots (2n - 1), \forall n \in \mathbb{N}$

Solusi:

$$(n + 1)(n + 2) \dots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots (2n - 1)$$

$$(n + 1)(n + 2) \dots 2n = 2^n \cdot (2n - 1)!!$$

a) Untuk $n = 1$

$$2 \cdot 1 = 2^1 \cdot (2 \cdot 1 - 1)$$

$$2 = 2$$

BENAR

b) Diasumsikan BENAR untuk $n = k$

$$(k + 1)(k + 2) \dots 2k = 2^k \cdot (2k - 1)!!$$

c) Akan dibuktikan BENAR untuk $n = k + 1$

$$((k + 1) + 1)((k + 1) + 2) \dots 2(k + 1) = 2^{(k+1)} \cdot (2(k + 1) - 1)!!$$

$$(k + 2)(k + 3) \dots (2k + 2) = 2^{(k+1)} \cdot (2k + 1)!!$$

$$(k + 1)(k + 2)(k + 3) \dots (2k) \frac{(2k + 1)(2k + 2)}{(k + 1)} = 2^{(k+1)} \cdot (2k + 1)!!$$

$$2^k \cdot (2k - 1)!! \frac{(2k + 1)(2k + 2)}{(k + 1)} = 2^{(k+1)} \cdot (2k + 1)!!$$

$$2^k \cdot (2k - 1)!! \frac{2(2k + 1)(k + 1)}{(k + 1)} = 2^{(k+1)} \cdot (2k + 1)!!$$

$$2^{k+1} \cdot (2k - 1)!! (2k + 1) = 2^{(k+1)} \cdot (2k + 1)!!$$

$$2^{(k+1)} \cdot (2k + 1)!! = 2^{(k+1)} \cdot (2k + 1)!!$$

Terbukti!

BENAR untuk $n = k + 1$

Terbukti $(n + 1)(n + 2) \dots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots (2n - 1), \forall n \in \mathbb{N}$