MPF1204, FISIKA KUANTUM (3 SKS)
Program Studi S2 Pendidikan Fisika

FAKULTAS KEGURUAN DAN ILMU PENDIDIKAN
UNIVERSITAS SEBELAS MARET (UNS) SURAKARTA

e Learning :

Eigen Function of Angular Momentum in The Spherical Coordinates and Hamiltonian Comuttator

Pertemuan ke-13 :
Selasa, 12 Mei 2020 TM online Pk. 08.30 wib

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1. Eigen Function of Angular Momentum

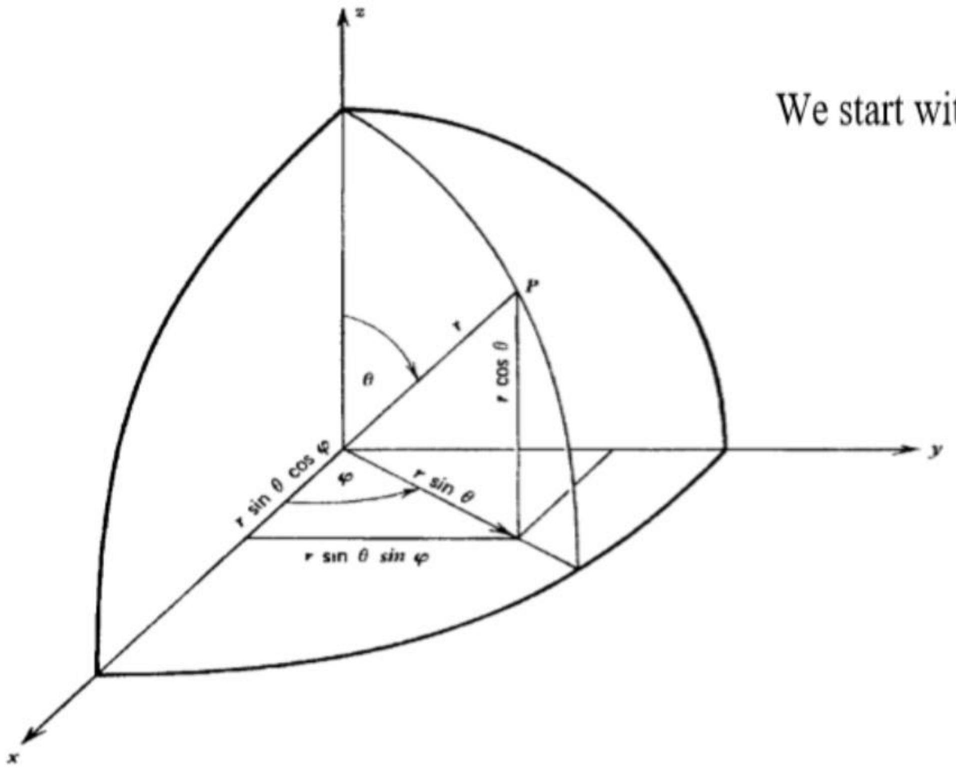

Figure 7-2 The definition of the spherical coordinates used in the text and the relation between the cartesian coordinates (x, y, z) and the spherical coordinates (r, θ, φ).

We start with spherical coordinates, as defined in Fig. 7-2. We have

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned} \tag{7B-1}$$

Eigen Function of Angular Momentum

From this it follows that

$$\begin{aligned} dx &= \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi \\ dy &= \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta \end{aligned} \tag{7B-2}$$

These may be solved to give

$$\begin{aligned} dr &= \sin\theta\cos\varphi\, dx + \sin\theta\sin\varphi\, dy + \cos\theta\, dz \\ d\theta &= \frac{1}{r}(\cos\theta\cos\varphi\, dx + \cos\theta\sin\varphi\, dy - \sin\theta\, dz) \\ d\varphi &= \frac{1}{r\sin\theta}(-\sin\varphi\, dx + \cos\varphi\, dy) \end{aligned} \tag{7B-3}$$

Eigen Function of Angular Momentum

With the help of these we obtain

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

$$= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \qquad (7B\text{-}4)$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

We may use these to calculate the angular momentum operators in terms of the spherical angles. We have

$$\mathbf{L} = \mathbf{r}\times\mathbf{p}_{\text{op}} = \frac{\hbar}{i}(\mathbf{r}\times\nabla) \qquad (7B\text{-}5)$$

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Eigen Function of Angular Momentum

$$= \frac{\hbar}{i}\left[r \sin\theta \cos\varphi\left(\sin\theta \sin\varphi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \sin\varphi \frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left. - r\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \qquad (7B\text{-}6)$$

Similarly, we construct

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= \frac{\hbar}{i}\left[r\sin\theta\sin\varphi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\right.$$

$$\left. - r\cos\theta\left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right] \qquad (7B\text{-}7)$$

$$= \frac{\hbar}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

Eigen Function of Angular Momentum

and

$$
\begin{aligned}
L_y &= \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\
&= \frac{\hbar}{i}\left[r\cos\theta\left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right. \\
&\qquad \left. - r\sin\theta\cos\varphi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\right] \\
&= \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)
\end{aligned}
\qquad (7B\text{-}8)
$$

Eigen Function of Angular Momentum

It is fairly straightforward to calculate

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$$

$$-\hbar^2\left[\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right.$$

$$\left.+\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{\partial^2}{\partial\varphi^2}\right]$$

We leave it to the reader to do the algebra. The final result is

$$\mathbf{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \qquad (7B\text{-}9)$$

Which is just the expression in eq. (7-13).

The equation

$$\mathbf{L}^2 Y(\theta, \varphi) = \hbar^2\lambda Y(\theta, \varphi)$$

2. Hamiltonian and Angular momentum Comuttator

Consider the special case of a rotation through an angle θ about the z-axis: With

$$x' = x\ \cos\ \theta - y\ \sin\ \theta$$
$$y' = x\ \sin\ \theta + y\ \cos\ \theta \qquad (7A\text{-}1)$$

it is easy to see that

$$r' = (x'^2 + y'^2 + z'^2)^{1/2} = (x^2 + y^2 + z^2)^{1/2} = r \qquad (7A\text{-}2)$$

and

$$\left(\frac{\partial}{\partial x'}\right)^2 + \left(\frac{\partial}{\partial y'}\right)^2 = \left(\cos\ \theta\frac{\partial}{\partial x} - \sin\ \theta\frac{\partial}{\partial y}\right)^2 + \left(\sin\ \theta\frac{\partial}{\partial x} + \cos\ \theta\frac{\partial}{\partial y}\right)^2$$
$$= \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \qquad (7A\text{-}3)$$

Hamiltonian and Angular momentum Comutator

Since the Hamiltonian has an invariance property, we expect a conservation law, as we saw in the case of parity. To identify the operators that commute with H, let us consider an infinitesimal rotation about the z-axis. Keeping terms of order θ only so that

$$\begin{aligned} x' &= x - \theta y \\ y' &= y + \theta x \end{aligned} \tag{7A-4}$$

we require that

$$Hu_E(x - \theta y, y + \theta x, z) = Eu_E(x - \theta y, y + \theta x, z) \tag{7A-5}$$

If we expand this to first order in θ and subtract from it

$$Hu_E(x, y, z) = Eu_E(x, y, z) \tag{7A-6}$$

we obtain from the term linear in θ

$$H\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)u_E(x, y, z) = E\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)u_E(x, y, z) \tag{7A-7}$$

Hamiltonian and Angular momentum Comutator

The right side of this equation may be written as

$$\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)E\, u_E(x, y, z) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)H\, u_E(x, y, z) \qquad (7A\text{-}8)$$

If we define

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = xp_y - yp_x \qquad (7A\text{-}9)$$

then (7A-7) and (7A-8) together read

$$(HL_z - L_z H)u_E(x, y, z) = 0$$

Since the $u_E(\mathbf{r})$ form a complete set, this implies the operator relation

$$[H, L_z] = 0 \qquad (7A\text{-}10)$$

Hamiltonian and Angular momentum Comutator

holds. L_z is the z-component of the operator

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad (7A\text{-}11)$$

which is the angular momentum. Had we taken rotations about the x- and y-axis, we would have found, in addition, that

$$[H, L_x] = 0$$
$$[H, L_y] = 0 \qquad (7A\text{-}12)$$

Thus the three components of the angular momentum operators commute with the Hamiltonian; that is, the angular momentum is a constant of the motion. This parallels the classical result that central forces imply conservation of the angular momentum.

Exercises :

1. Calculate $\langle l, m_1|L_x|l, m_2\rangle$ and $\langle l, m_1|L_y|l, m_2\rangle$.

2. Calculate $\langle l, m_1|L_x^2|l, m_2\rangle$ and $\langle l, m_1|L_y^2|l, m_2\rangle$. (*Hint:* Things will be easier with the use of $L_\pm$, $L_\pm^2$, and so on.)

Thank you