MPF1204, FISIKA KUANTUM (3 SKS)
Program Studi S2 Pendidikan Fisika

FAKULTAS KEGURUAN DAN ILMU PENDIDIKAN
UNIVERSITAS SEBELAS MARET (UNS) SURAKARTA

e Learning :

Angular Momentum :
Commutation Ralations and Operators for Angular Momentum

Pertemuan ke-12 :
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1. Introduction

In our discussion of the Schrödinger equation in three dimensions we will need to deal with the kinetic energy in three dimensions, which has the form

$$K = \frac{\mathbf{p}^2}{2\mu} \tag{7-1}$$

for problems in which the effective mass of the particle is μ. Just as in classical mechanics, there is a close connection between $\mathbf{p}^2$ and the square of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We shall take over this expression for the angular momentum into quantum mechanics, with the recognition that $\mathbf{r}$ and $\mathbf{p}$ are to be treated as operators. We will also see (Supplement 7-A [www.wiley.com/college/gasiorowicz]) that when the potential in a three-dimensional Schrödinger equation

$$H\psi(\mathbf{r}) = \left(\frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r})\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{7-2}$$

The conservation of angular momentum implies the operator equation

$$\frac{d\mathbf{L}}{dt} = 0 \tag{7-4}$$

This, as seen in Chapter 6, is equivalent to

$$[H, \mathbf{L}] = 0 \tag{7-5}$$

2. The Angular Momentum Commutation Ralations

We might be tempted to look for simultaneous eigenfunctions of H and all three components of **L**. This, as we have seen in Chapter 5, is only possible if all four operators commute with each other. To proceed we must check whether *all* of the operators L_x, L_y, and L_z commute with each other, as well as with H. In fact, different components of the angular momentum *do not* commute with each other. For example, paying particular attention to the ordering of the operators, we get, using $\mathbf{L} = \mathbf{r} \times \mathbf{p}$,

$$\begin{aligned}[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\ &= y[p_z, z]p_x + x[z, p_z]p_y \\ &= \frac{\hbar}{i}(yp_x - xp_y) = i\hbar L_z\end{aligned} \qquad (7\text{-}6a)$$

Similarly we can show that

$$[L_y, L_z] = i\hbar L_x \qquad (7\text{-}6b)$$

and

$$[L_z, L_x] = i\hbar L_y \qquad (7\text{-}6c)$$

The Angular Momentum Commutation Ralations…

It is true that each of the components of angular momentum commutes with $\mathbf{L}^2$. For example,

$$\begin{aligned}[L_z, L_x^2 + L_y^2 + L_z^2] &= L_y[L_z, L_y] + [L_z, L_y]L_y + L_x[L_z, L_x] + [L_z, L_x]L_x \\ &= -i\hbar L_y L_x - i\hbar L_x L_y + i\hbar L_x L_y + i\hbar L_y L_x = 0\end{aligned} \tag{7-7}$$

We can see that as a consequence of these commutation relations, only one component of $\mathbf{L}$ may be chosen with H and $\mathbf{L}^2$ to form a simultaneously commuting set. To show this let us assume that we have a set of eigenfunctions that are simultaneous eigenfunctions of all three components of $\mathbf{L}$. Let us assume that

$$L_x|u\rangle = l_1|u\rangle$$

and

$$L_y|u\rangle = l_2|u\rangle$$

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which implies that $L_xL_y|u\rangle = l_1l_2|u\rangle$ and $L_yL_x|u\rangle = l_2l_1|u\rangle$. As a consequence of (7-6a) this means that $L_z|u\rangle = 0$. This, however implies that

$$l_2|u\rangle = L_y|u\rangle = \frac{1}{i\hbar}[L_z, L_x]|u\rangle = \frac{1}{i\hbar}L_zl_1|u\rangle = 0$$

Similarly, we can show that $l_1|u\rangle = 0$. This means that only for $\mathbf{L} = 0$ can we have simultaneous eigenfunction for all three components of the angular momentum.

There is nothing to keep us from picking just one component of **L** as part of the commuting set. Conventionally the choice is L_z, but there is nothing special about this choice. We thus will deal with simultaneous eigenfunctions of $\mathbf{L}^2$ and L_z. We will denote the eigenkets by $|l, m\rangle$. Our starting point is thus the set of equations

$$\begin{aligned} \mathbf{L}^2|l, m\rangle &= \hbar^2 l(l+1)|l, m\rangle \\ L_z|l, m\rangle &= \hbar m|l, m\rangle \end{aligned} \tag{7-8}$$

2. Lowering Operators for Angular Momentum

Our starting point is (7-8), together with the angular momentum commutation relations (7-6) and the orthonormality relation

$$\langle l', m' | l, m \rangle = \delta_{ll'}\delta_{mm'} \tag{7-9}$$

It will prove convenient to introduce the operators

$$L_\pm = L_x \pm iL_y \tag{7-10}$$

These obey the commutation relations

$$\begin{aligned} [L_+, L_-] &= [L_x + iL_y, L_x - iL_y] = (-2i)[L_x, L_y] \\ &= 2\hbar L_z \end{aligned} \tag{7-11}$$

and

$$\begin{aligned} [L_z, L_\pm] &= [L_z, L_x \pm iL_y] = i\hbar L_y \mp i(i\hbar L_x) = \pm\hbar(L_x \pm iL_y) \\ &= \pm\hbar L_\pm \end{aligned} \tag{7-12}$$

Lowering Operators for Angular Momentum…

It is also obvious that

$$[\mathbf{L}^2, L_\pm] = 0 \qquad (7\text{-}13)$$

Furthermore, we have

$$\begin{aligned} L_+L_- &= (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y] \\ &= \mathbf{L}^2 - L_z^2 + \hbar L_z \end{aligned} \qquad (7\text{-}14)$$

and similarly,

$$L_-L_+ = \mathbf{L}^2 - L_z^2 - \hbar L_z \qquad (7\text{-}15)$$

Thus

$$L_+L_- + L_z^2 - \hbar L_z = L_-L_+ + L_z^2 + \hbar L_z = \mathbf{L}^2 \qquad (7\text{-}16)$$

We now note that $\langle l, m|L_x^2|l, m\rangle = \langle L_x(l, m)|L_x(l, m)\rangle \geq 0$ and by extension $\langle l, m|\mathbf{L}^2|l, m\rangle \geq 0$. This implies that $l(l + 1) \geq 0$. From this it follows that $l \geq 0$. (The alternative that $l \leq -1$ we reject, since we would then call $l + 1 = -l'$ and get $l' \geq 0$.) First, we note that

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$$\mathbf{L}^2 L_\pm|l, m\rangle = L_\pm \mathbf{L}^2|l, m\rangle = \hbar^2 l(l + 1)L_\pm|l, m\rangle \qquad (7\text{-}17)$$

This means that $L_\pm|l, m\rangle$ is an eigenstate of $\mathbf{L}^2$ with the eigenvalue characterized by l. On the other hand,

$$L_z L_+|l, m\rangle = (L_+L_z + \hbar L_+)|l, m\rangle = \hbar(m + 1)L_+|l, m\rangle \qquad (7\text{-}18)$$

and similarly

$$L_z L_-|l, m\rangle = \hbar(m - 1)L_-|l, m\rangle \qquad (7\text{-}19)$$

These equations imply that $L_+|l, m\rangle$ is an eigenstate of L_z with the m value raised by unity, and $L_-|l, m\rangle$ is an eigenstate of L_z with m value lowered by unity. We therefore call $L_\pm$ *raising* and *lowering* operators, respectively. We may write

$$\begin{aligned} L_+|l, m\rangle &= C_+(l, m)|l, m + 1\rangle \\ L_-|l, m\rangle &= C_-(l, m)|l, m - 1\rangle \end{aligned} \qquad (7\text{-}20)$$

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The conjugate relation of the first of the above is

$$\langle l, m | L_- = \langle l, m+1 | C_+^*(l, m) \qquad (7\text{-}21)$$

Multiplying this with the first of (7-20) yields

$$\begin{aligned} |C_+(l, m)|^2 \langle l, m+1 | l, m+1 \rangle &= \langle l, m | L_- L_+ | l, m \rangle = \langle l, m | \mathbf{L}^2 - L_z^2 - \hbar L_z | l, m \rangle \\ &= \hbar^2 [l(l+1) - m^2 - m] \qquad (7\text{-}22) \\ &= \hbar^2 [(l-m)(l+m+1)] \end{aligned}$$

Thus

$$C_+(l, m) = \hbar \sqrt{(l-m)(l+m+1)} \qquad (7\text{-}23)$$

and, similarly

$$C_-(l, m) = \hbar \sqrt{l(l+1) - m(m-1)} = \hbar \sqrt{(l+m)(l-m+1)} \qquad (7\text{-}24)$$

It follows from

$$\langle L_\pm(l, m) | L_\pm(l, m) \rangle \geq 0$$

Lowering Operators for Angular Momentum…

that

$$\langle L_\pm(l,m)|L_\pm(l,m)\rangle = \langle l,m|L_\mp L_\pm|l,m\rangle$$
$$= \langle l,m|\mathbf{L}^2 - L_z^2 \pm \hbar L_z|l,m\rangle \tag{7-25}$$
$$= \hbar^2[l(l+1) - m(m\mp 1)] \geq 0$$

This implies that both

$$l(l+1) \geq m(m+1)$$
$$l(l+1) \geq m(m-1) \tag{7-26}$$

are true. Since $l \geq 0$, it follows from the above that

$$-l \leq m \leq l \tag{7-27}$$

Let us assume that the minimum value of m is m_{min}. This means that we cannot lower the m-value any further, and thus

$$L_-|l, m_{min}\rangle = 0 \tag{7-28}$$

We can see, in a number of ways (by looking at $C_-(l,m)$ for example), that

$$m_{min} = -l \tag{7-29}$$

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Similarly, the maximum value of m, denoted by m_{max} is such that

$$L_+ |l, m_{max}\rangle = 0 \qquad (7\text{-}30)$$

and

$$m_{max} = l \qquad (7\text{-}31)$$

Since the maximum value is to be reached from the minimum value by unit steps (repeated application of L_+), we find, as seen in Fig. 7-1, that there are $(2l + 1)$ steps. This implies that $(2l + 1)$ is an integer, and m can take on the values

$$m = -l, -l + 1, -l + 2, \ldots, l - 1, l \qquad (7\text{-}32)$$

The possibility that l is half-integral, $l = 1/2, 3/2, 5/2, \ldots$, will be discussed in Chapter 10, when we discuss *spin*. Until then, we restrict ourselves to *integer values of l*.

Lowering Operators for Angular Momentum…

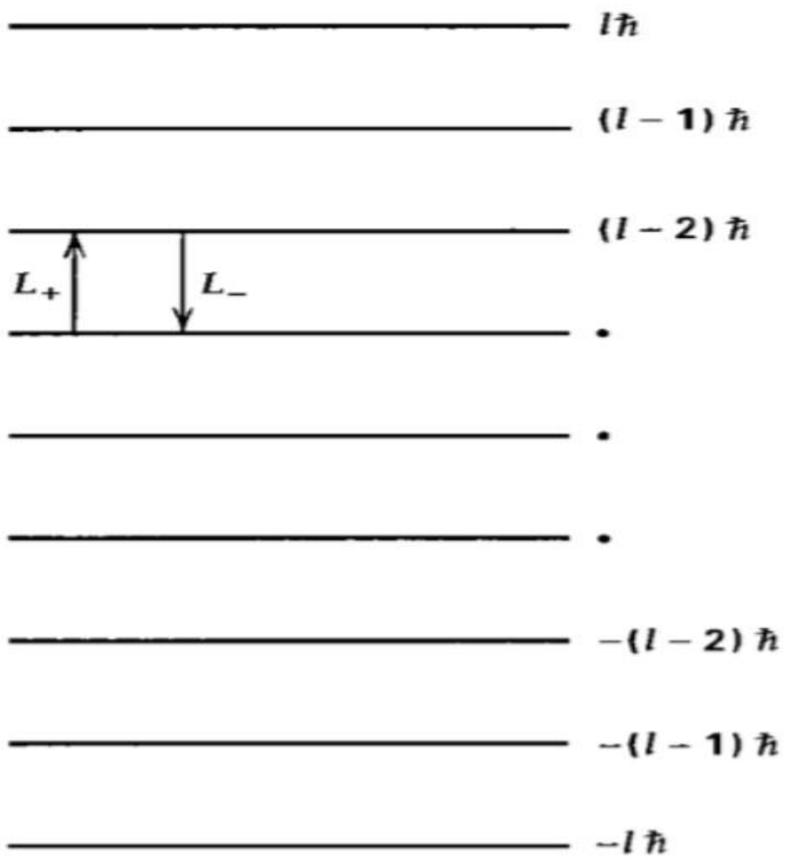

Figure 7-1 Spectrum of the operator L_z for a given value of l.

Exercises: (at paper),

1. Calculate $\langle l, m_1|L_x|l, m_2\rangle$ and $\langle l, m_1|L_y|l, m_2\rangle$.

2. Calculate the commutators $[x, L_x]$, $[y, L_x]$, $[z, L_x]$, $[x, L_y]$, $[y, L_y]$, $[z, L_y]$. Do you detect a pattern that will allow you to state the commutators of x, y, z with L_z?

3. Express the spherical harmonics for $l = 0, 1, 2$ in terms of x, y, z.

Thank you