Module 6: A.C. Electrical Machines for Hybrid and Electric Vehicles

Lecture 17: Induction motors, their configurations and optimization for HEV/EVs

Fundamentals of Electrical Machines

Introduction

The topics covered in this chapter are as follows:

- Electrical Machines in EVs and HEVs
- Physical Concepts of Torque production
- Why Should the Number of Poles on Stator Equal to the Number of Poles on Rotor
- How Continuous Torque is Produced by a Motor
- Rotating Magnetic Field
- How to Create the Second Magnetic Field
- Electrical and Mechanical Angle

Electrical Machines in EVs and HEVs

Vehicle propulsion has specific requirements that distinguish stationary and onboard motors. Every kilogram onboard the vehicle represents an increase in structural load. This increase structural load results in lower efficiency due to increase in the friction that the vehicle has to overcome. Higher efficiency is equivalent to a reduction in energy demand and hence, reduced battery weight.

The fundamental requirement for traction motors used in EVs is to generate propulsion torque over a wide speed range. These motors have intrinsically neither nominal speed nor nominal power. The power rating mentioned in the catalog and on the name plate of the motor corresponds to the maximum power that the drive can deliver. Two most commonly used motors in EV propulsion are Permanent Magnet (PM) Motors and Induction Motors (IM). These two motors will be investigated in detail in the coming lectures. However, before going into the details of these machines some basic fundamentals of electrical machines, such as torque production, are discussed in this chapter.

Physical Concepts of Torque Production

In **Figure 1a** a stator with 2 poles and a cylindrical rotor with a coil are shown. When only the stator coils are energized, stator magnetic flux is set up as shown in **Figure 1a**. The magnetic field for case when only the rotor coil is energized is shown in **Figure 1b**. In case when both the stator and rotor coils are energized, the magnetic resultant magnetic field is shown in **Figure 1c**. Since in this case the magnetic flux lines behave like stretched band, the rotor conductor experiences a torque in the direction shown in **Figure 1c**. From **Figure 1c** it can be seen that the stator ***S*** pole attracts the rotor ***N*** pole and repels the rotor ***S*** pole, resulting in clockwise torque. Similarly stator ***N*** pole attracts rotor ***S*** pole and repels rotor ***N*** pole, resulting again in clockwise torque.

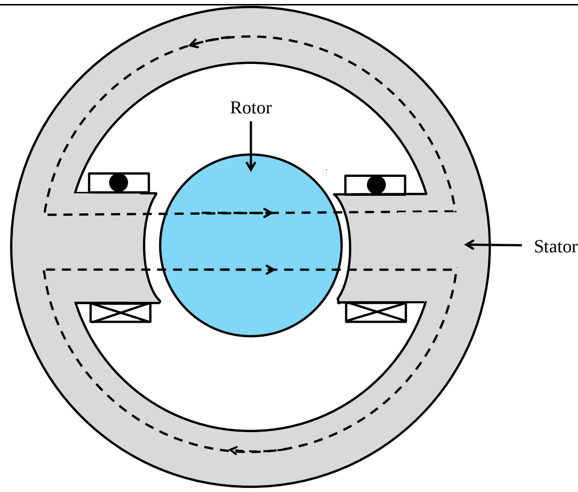

Figure 1a: Magnetic field when only stator is energised

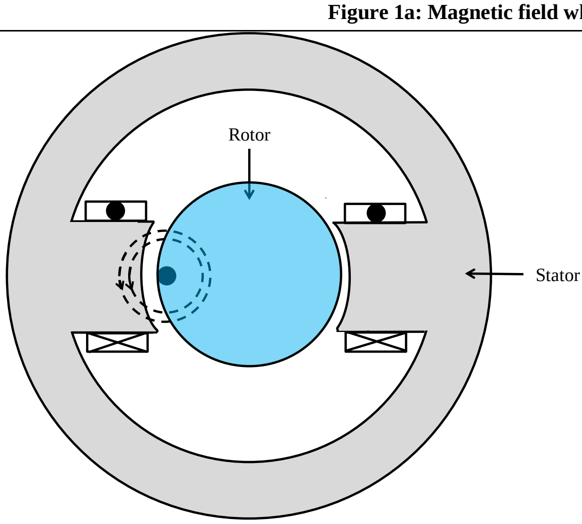

Figure 1b: Magnetic field when only rotor is energised

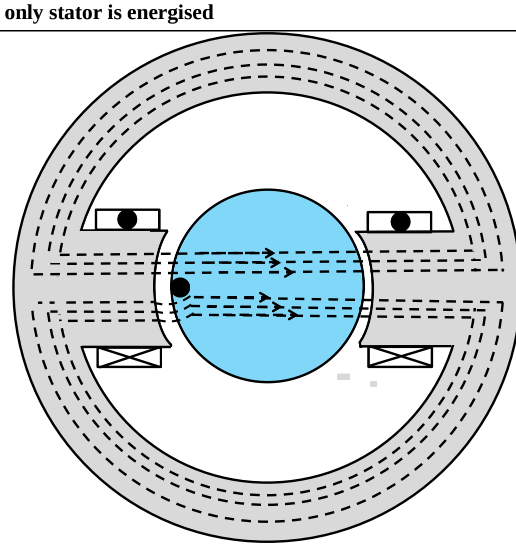

Figure 1c: Magnetic field when both stator and rotor are energised

The total torque is shown in **Figure 1c**. This torque is developed due to the interaction of stator and rotor magnetic fields and hence is known as ***interaction torque*** or ***electromagnetic torque***. The magnitude of the electromagnetic torque (T_{em}) or interaction torque is given by

$$T_{em} \propto (H_s)(H_r)\sin\delta \qquad (1)$$

where

H_s is the magnetic field created by current in the stator winding

H_r is the magnetic field created by current in the rotor winding

δ is the angle between stator and rotor magnetic field

Another configuration of the motor, with the flux lines, is shown in **Figure 2a**. Since the magnetic flux has a tendency to follow a minimum reluctance path or has a tendency to shorten its flux path, the rotor experiences an anti-clockwise torque. From **Figure 2a** it can be seen that the flux lines will have a tendency to align the rotor so that the reluctance encountered by them is reduced. The least reluctance position of the rotor is shown in **Figure 2b**.

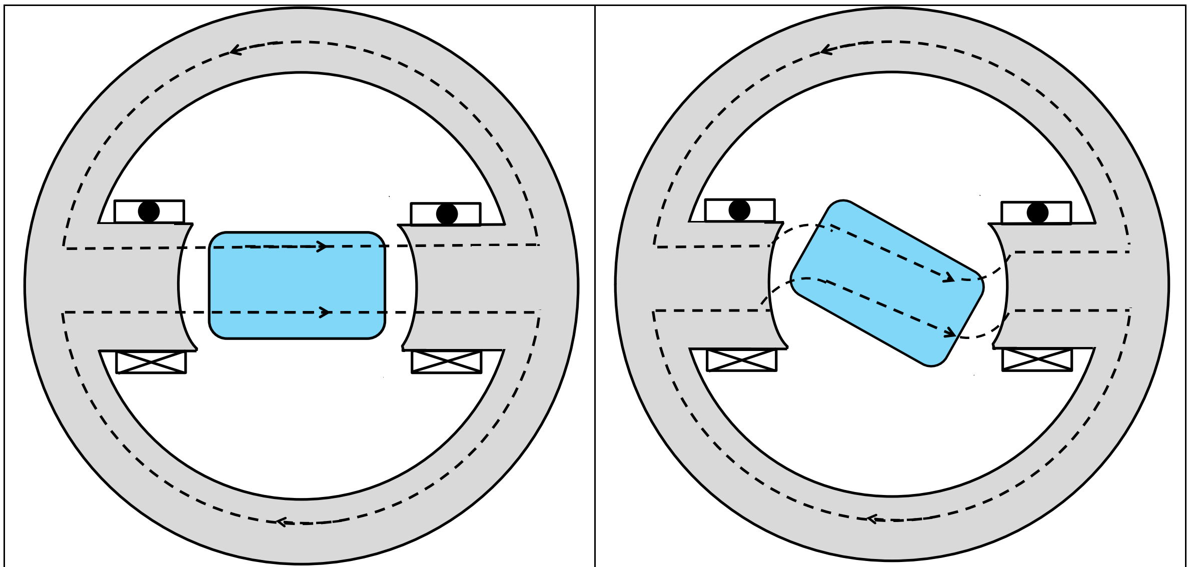

Figure 2a: Magnetic field when rotor axis is aligned with stator poles

Figure 2b: Magnetic field when rotor axis is not aligned with stator poles

To realign the rotor from the position shown in **Figure 2a** to position shown in **Figure 2b**, a torque is exerted by the flux lines on the rotor. This torque is known as the ***reluctance*** or ***alignment torque***.

Why Should the Number of Poles on Stator Equal to the Number of Poles on Rotor?

In the previous section it has been shown that to produce electromagnetic torque, the magnetic field produced by the stator has to interact with the magnetic field produced by the rotor. However, ***if the number of poles producing the stator magnetic field is not equal to the number of rotor poles producing the rotor magnetic field, then the net torque produced by the motor will be zero***. This is illustrated by the motor configuration shown in **Figure 3**. In this motor the stator has two poles (N_s, S_s) and the rotor has four poles ($N_{r1}, S_{r1}, N_{r2}, S_{r2}$). The angle between the stator poles is 180^o and the angle between the rotor poles is 90^o. From the arrangement shown in **Figure 3** it can be seen that the angle between N_{r1} and N_s is equal to the angle between N_{r2} and S. Hence, a repulsive force exists between N_{r1} and N_s in clockwise direction and an attractive force exists between N_{r2} and S in the anticlockwise direction. Both, the attractive and repulsive forces are of same magnitude and the resultant of these forces is zero.

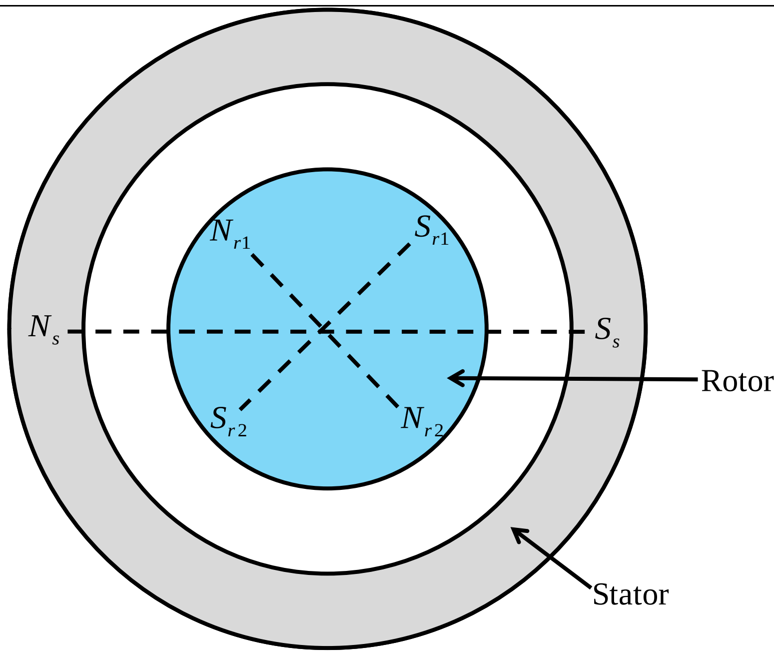

Figure 3: Configuration of a motor with unequal number of stator and rotor poles

Now consider the pole pairs (N_s, S_{r2}) and (S_s, S_{r1}), the angle between the pole pairs is same. Hence, the force of attraction between N_s and S_{r2} is same as the force of repulsion between S_s and S_{r1} and thus, the resultant force acting on the rotor is zero. Therefore, in this case no electromagnetic torque is developed.

From the above discussion it can be seen that the resultant electromagnetic torque developed due to two stator poles and 4 rotor poles is zero. This leads to the conclusion that ***in all rotating electric machines, the number of rotor poles should be equal to number of stator poles for electromagnetic torque to be produced***.

How Continuous Torque is Produced by a Motor

In the previous section it has been seen that to produce electromagnetic torque, following two conditions have to be satisfied:

- Both stator and rotor must produce magnetic field
- The number of magnetic poles producing the stator magnetic field must be same as the number of magnetic poles producing the rotor magnetic field.

Now an important question that arises is ***how to create continuous magnetic torque***? To produce continuous torque the magnetic field of the stator should rotate continuously. As a result, the rotor's magnetic field will chase the stator's magnetic field and this result in production of continuous torque. This phenomenon is illustrated in **Figures 4a-4d**. In **Figure 4** a two pole machine is depicted and the rotors magnetic field is created by the permanent magnets. It is assumed that the stator's magnetic field rotates at a speed of 60 revolutions per minute (60 rpm) which is equivalent 1 revolution per second (1rps). To start the analysis it is assumed that at time $t = 0$, the stator's magnetic field axis aligns itself with the $x-$axis and the rotor's magnetic field axis makes an angle δ with the stator's magnetic axis (**Figure 4a**). At time $t = 0.25s$, the stator's magnetic field moves by 90^{o} and the rotor's magnetic field chases the stator's field and aligns as shown in **Figure 4b**. Similarly the locations of the magnetic field axis at time $t = 0.5s$ and $t = 0.75s$ are shown in **Figures 4c** and **Figure 4d**.

From the above discussion and observing **Figure 4** the following conclusions can be drawn:

- The rotor's magnetic field chases the stator's magnetic field.
- The angle (δ) between the stator's magnetic axis and the rotor's magnetic axis remains constant. Hence, the rotor's speed of rotation is same as that of the stator's magnetic field.

However, an important question that still remains unanswered is ***How to create a rotating magnetic field?***

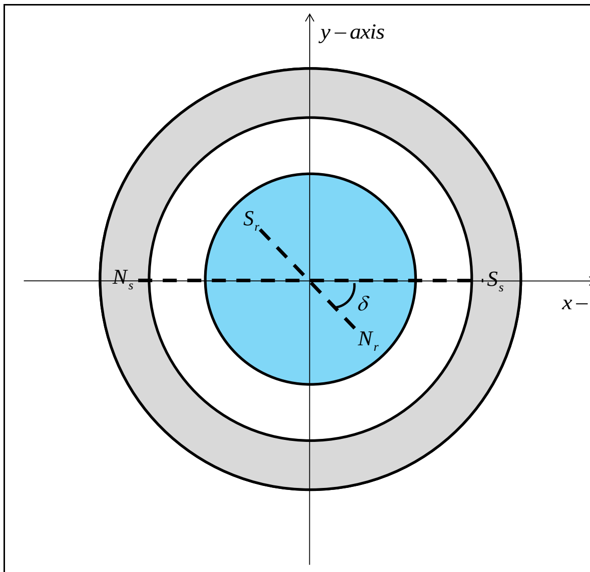

Figure 4a: Stator's magnetic field at time $t = 0$

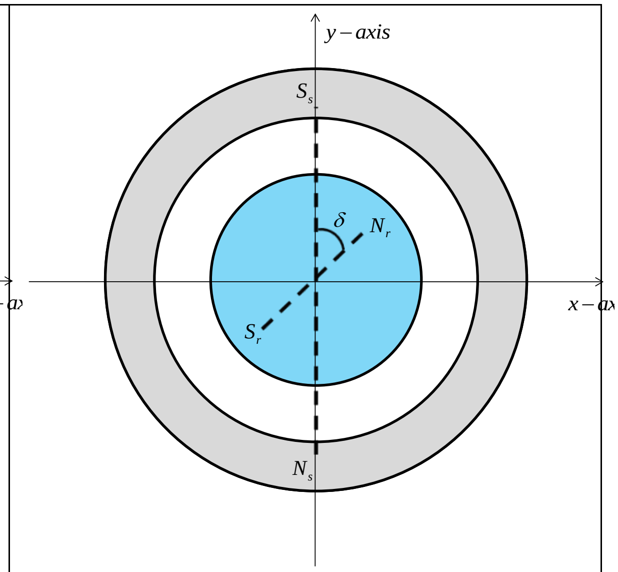

Figure 4b: Stator's magnetic field at time $t = 0.25$

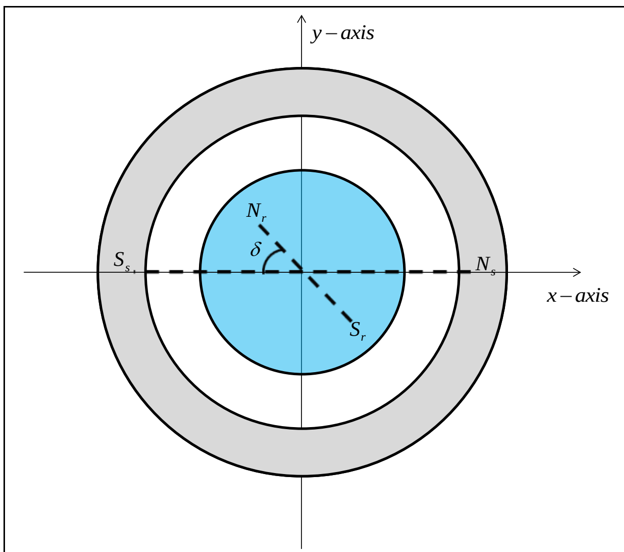

Figure 4c: Stator's magnetic field at time $t = 0.5$

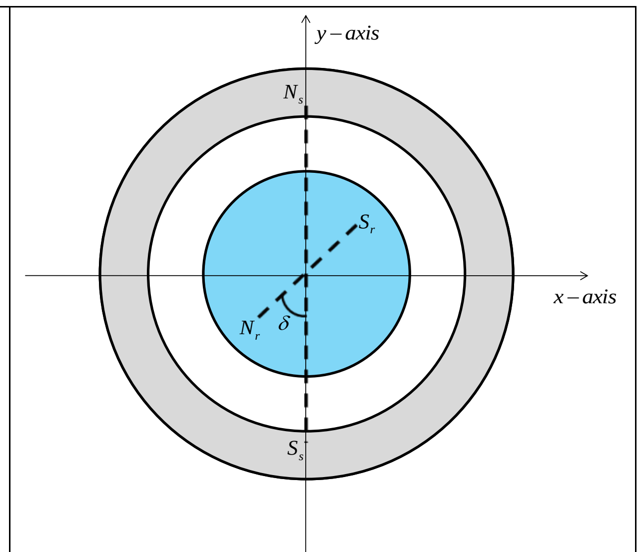

Figure 4d: Stator's magnetic field at time $t = 0.75$

Rotating Magnetic Field

To understand the rotations of magnetic field consider a 2-pole 3-phase stator as shown in **Figure 5a**. The three phase windings ***a, b*** and ***c*** are represented by three coils aa', bb' and cc'. A current in phase ***a*** winding establishes magnetic flux directed along the magnetic axis of coil aa'. Similarly, the currents in phase ***b*** and ***c*** windings will create fluxes directed along the magnetic axes of coils bb' and cc' respectively. The three phase currents flowing the winding is shown in **Figure 5a**. At time instant **1**, the currents of each phase are

$$i_a = I_m \;;\; i_b = -\frac{I_m}{2} \;;\; i_c = -\frac{I_m}{2}$$

where (2)

$I_m =$ maximum value of the current

Since, i_b and i_c are negative, crosses must be shown in coil-sides b' and c' and dots in the coil sides b and c. The right hand thumb rule gives the flux distribution as shown in **Figure 5b**. In **Figure 5b** and the following figures, the thicker line indicates higher magnitude to flux. The

At instant **2**, the currents are

$$i_a = \frac{I_m}{2} \;;\; i_b = \frac{I_m}{2} \;;\; i_c = -I_m \tag{3}$$

The magnetic flux distribution created by the currents at instant **2** is shown in **Figure 5c**. Eventually at instant **3**, the currents are

$$i_a = -\frac{I_m}{2} \;;\; i_b = I_m \;;\; i_c = -\frac{I_m}{2} \tag{4}$$

The magnetic flux distribution created by the phase currents given by **equation 4** is shown in **Figure 5d**. From **Figure 5b** to **5c** it can be seen that the 2 poles produced by the resultant flux are seen to have turned 60°. At other instants of time, i.e. as time elapses, the two poles rotate further. In this manner a rotating magnetic field is produced. The space angle traversed by a rotating flux is equal to the time angle traversed by currents.

After having discussed the production of rotating magnetic field, an important issue that still remains unresolved is: ***How to create the second magnetic field that will follow the rotating magnetic field created by the stator?*** This question is answered in following section.

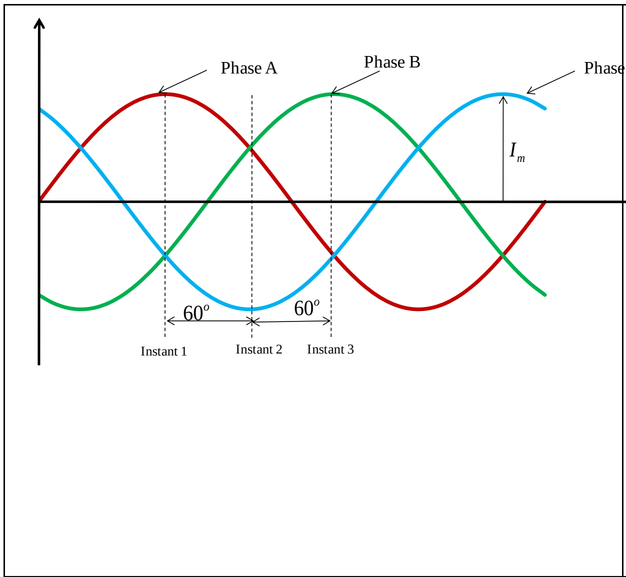

Figure 5a: Three phase currents given to stator windings

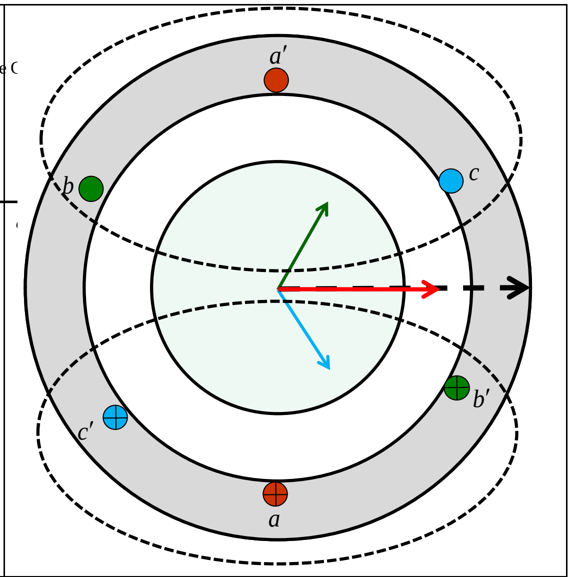

Figure 5b: Magnetic flux at *instant 1*

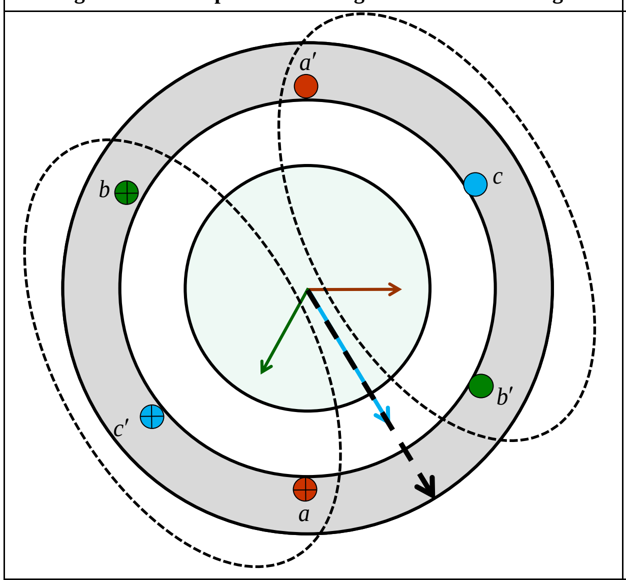

Figure 5c: Magnetic flux at *instant 2*

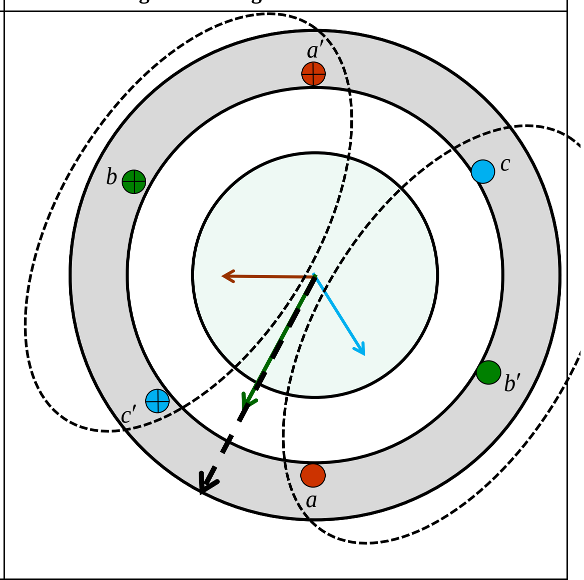

Figure 5d: Magnetic flux at *instant 3*

How to Create the Second Magnetic Field

From **equation 1** it can be seen that to produce torque two magnetic fields are required. The rotating magnetic field created by the stator has been discussed in the previous section and this section deals with the generation of rotor magnetic field. There are multiple ways to produce the rotor magnetic field namely:

- Having windings on the rotor and exciting then with dc current to produce magnetic field (known as ***Synchronous Machines***).
- Having permanent magnets on the rotor to produce the rotor magnetic field (known as ***Permanent Magnet Synchronous Machines***).
- Utilize the Faradays law of induction to induce electromotive force (e.m.f) in the rotor coils. The induced e.m.f will result in flow of current through the rotor conductors and these currents will produce a magnetic field. These machines are known as ***Induction Machines*** or ***asynchronous machines***.

Synchronous Machines

The general configuration of synchronous machine is shown in **Figure 6**. It can be seen from **Figure 6** that the rotor has a coil (denoted by a dot and a cross) and through this coil a dc current flows. Due to this dc current a pair of magnetic poles is created. The stator windings also create two magnetic fields that rotate with time and hence, the rotor's magnetic poles chase the stator's magnetic field and in the process electromagnetic torque is produced. The speed of rotation of rotor depends on the speed with which the stator's field rotates and hence, these machines are known as ***synchronous machine***.

Permanent Magnet Synchronous Machines (PMSM)

In case of PMSM, the rotor field is created by permanent magnets rather than dc current passing through a coil (**Figure 7**). The principle of operation of PMSM is same as that of synchronous machine.

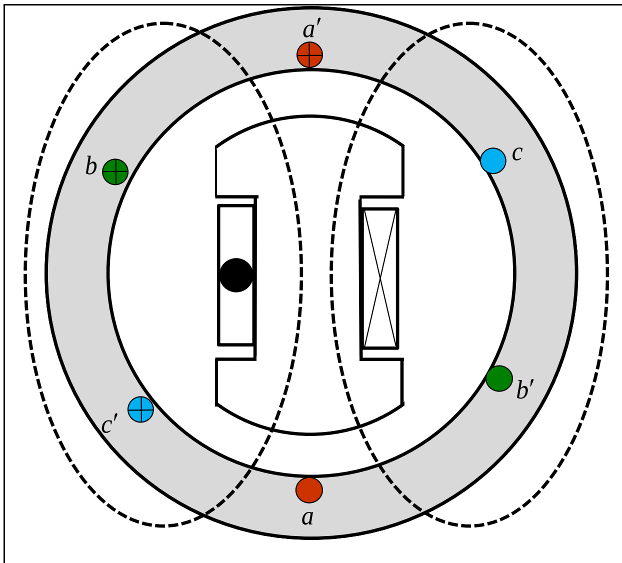

Figure 6: Synchronous machine

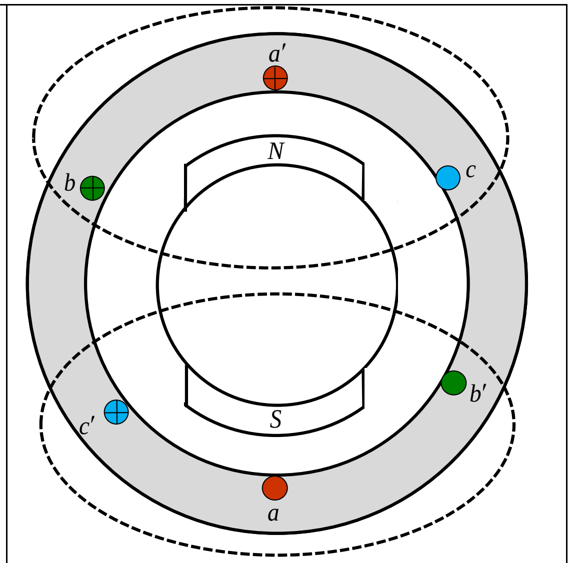

Figure 7: Permanent Magnet Synchronous machine

Induction Machine (IM)

Like synchronous machine, the stator winding of an induction machine is excited with alternating currents. In contrast to a synchronous machine in which a field winding on the rotor is excited with dc current, alternating currents flow in the rotor windings of an induction machine. In IM, alternating currents are applied to the stator windings and the rotor currents are produced by induction. The details of the working of the IMs are given in the following lectures.

After having discussed the general features of the electrical machines, the question that arises is***: how to analyse the machines?*** The analysis of electrical machines becomes simple by use of electrical equivalent circuits. The electrical equivalent circuits for the machines are discussed in the next section. One last concept that is relevant to electrical machines is principle of ***electrical*** and ***mechanical angle*** which is explained in the next section.

Electrical and Mechanical Angle

In **Figure 8**, it is assumed that the field winding is excited by a dc source and a coil rotates in the air gap at a uniform angular speed. When the conductor is aligned along $y-y'$ axis, the e.m.f induced is zero. Along $x-x'$ axis the induced e.m.f is maximum. In one revolution of the coil, the e.m.f induced is shown in **Figure 9**. If the same coil rotates in a 4 pole machine (**Figure 10**), excited by a dc source, the variation in the magnetic flux density and the induced e.m.f is shown in **Figure 11**. From **Figure 11** it can be seen that in one revolution of 360 mechanical degrees, 2 cycles of e.m.f (720 electrical degrees) are induced. The 720 electrical degrees in a 4 pole machine can be related to 360 mechanical degrees as follows

$$\begin{aligned} &720 \text{ electrical degrees} = \frac{4}{2}\times(360 \text{ mechanical degrees}) \\ &\Rightarrow \theta_{elec} = \frac{4}{2}\theta_{mech} \end{aligned} \tag{5}$$

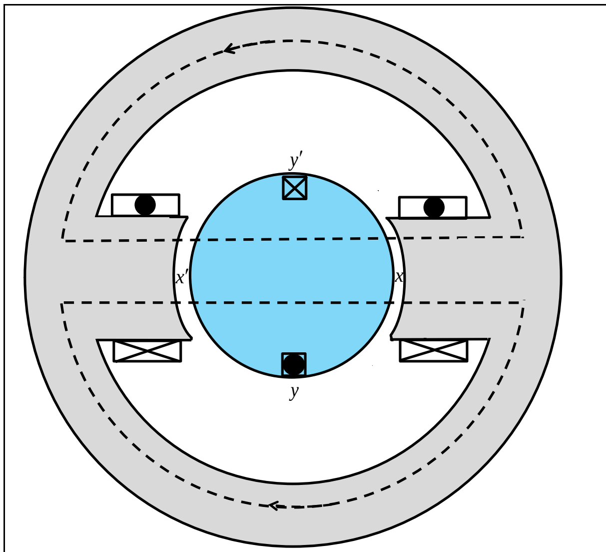

Figure 8: A two pole machine

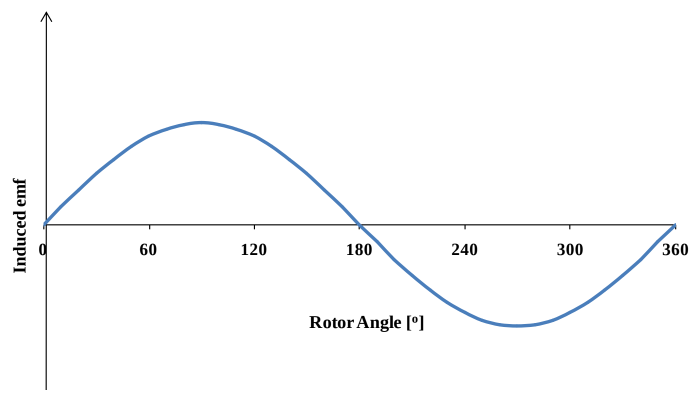

Figure 9: Induced e.m.f in the rotor coils of a two pole machine

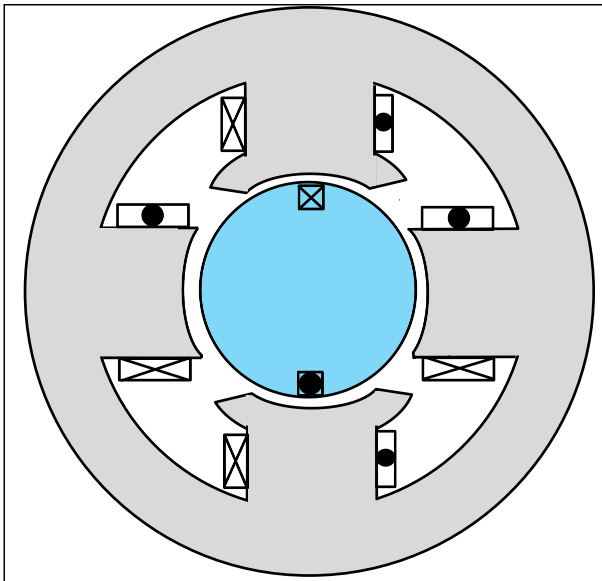

Figure 10: A four pole machine

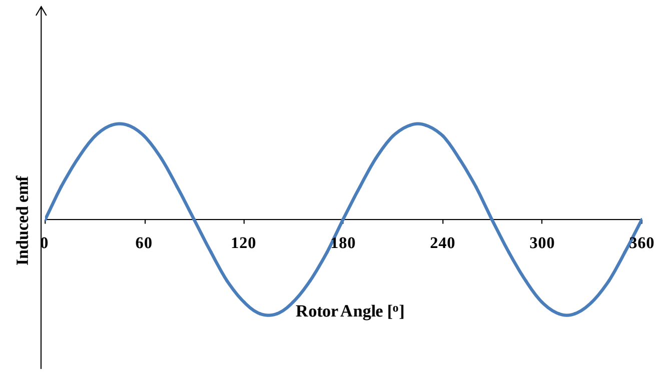

Figure11: Induced e.m.f in the rotor coils of a four pole machine

For a **P**-pole machine, **P/2** cycles of e.m.f will be generated in one revolution. Thus, for a **P** pole machine

$$\begin{aligned}&\theta_{elec} = \frac{P}{2}\theta_{mech}\\ &\Rightarrow \frac{d\theta_{elec}}{dt} = \frac{P}{2}\frac{d\theta_{mech}}{dt} \qquad (6)\\ &\Rightarrow \omega_{elec} = \frac{P}{2}\omega_{mech}\end{aligned}$$

In a 4 pole, in one revolution 2 cycles of e.m.f are generated. Hence, for a **P** pole machine, in one revolution **P/2** cycles are generated. For a **P**-pole machine, in one revolution per second, **P/2** cycles per second of e.m.f will be generated. Hence, for a **P** pole machine, in **n** revolutions per second $\frac{P}{2}\times n$ cycles/second are generated. The quantity cycles/second is the frequency ***f*** of the generated e.m.f. Hence,

$$f = \frac{P}{2}\times n \text{ Hertz} \Rightarrow f = \frac{PN}{120} \text{ Hertz} \qquad (7)$$

where

$N =$ the speed in rpm

Suggested Reading:

[1] M. G. Say, *The Performance and Design of Alternating Current Machines*, CBS Publishers, New Delhi

[2] S. J. Chapman, *Electric Machinery Fundamentals*, McGraw Hill, 2005

Lecture 18: Induction motor drives, their control and applications in EV/HEVs

Induction Motor for EV and HEV Application

Introduction

The topics covered in this chapter are as follows:

- Traction Motors
- Principle of Operation of Induction Motor (Mathematical Treatment)
- Principle of Operation of Induction Motor (Graphical Treatment)
- Fluxes and MMF in Induction Motor
- Rotor Action
- Rotor e.m.f and Equivalent Circuit
- Complete Equivalent Circuit
- Simplification of Equivalent Circuit
- Analysis of Equivalent Circuit
- Thevenin's Equivalent Circuit

Principles of Operation of Induction Motor (Mathematical Treatment)

In **Figure 1** a cross section of the stator of a three phase, two pole induction motor is shown. The stator consists of three blocks of iron spaced at 120° apart. The three coils are connected in Y and energized from a three phase system. When the stator windings are energized from a three phase system, the currents in the coils reach their maximum values at different instants. Since the three currents are displaced from each other by 120° electrical, their respective flux contributions will also be displaced by 120° electrical. Let a balanced three phase current be applied to the stator with the phase sequence ***A-B-C***

$$
\begin{aligned}
I_A &= I_m \cos \omega t \\
I_B &= I_m \cos\left(\omega t - \frac{2\pi}{3}\right) \\
I_C &= I_m \cos\left(\omega t - \frac{4\pi}{3}\right)
\end{aligned} \tag{1}
$$

The instantaneous flux produced by the stator will hence be

$$
\begin{aligned}
\phi_A &= \phi_m \cos \omega t \\
\phi_B &= \phi_m \cos\left(\omega t - \frac{2\pi}{3}\right) \\
\phi_C &= \phi_m \cos\left(\omega t - \frac{4\pi}{3}\right)
\end{aligned} \tag{2}
$$

The resultant flux at an angle θ from the axis of phase ***A*** is

$$\phi_T = \phi_A \cos(\theta) + \phi_B \cos(\theta - \frac{2\pi}{3}) + \phi_C \cos(\theta - \frac{4\pi}{3}) \tag{3}$$

Substituting **equation 2** into **equation 3** gives

$$\begin{aligned} &\phi_T = \phi_A \cos(\theta)\cos(\omega t) + \phi_B \cos(\theta - \frac{2\pi}{3})\cos(\omega t - \frac{2\pi}{3}) + \phi_C \cos(\theta - \frac{4\pi}{3})\cos(\omega t - \frac{4\pi}{3}) \\ &\Rightarrow \phi_T = \frac{3}{2}\phi_m \cos(\theta - \omega t) \end{aligned} \tag{4}$$

From **equation** 4 it can be seen that the resultant flux has amplitude of $1.5\phi_m$, is a sinusoidal function of angle θ and rotates in synchronism with the supply frequency. Hence, it is called a ***rotating field***.

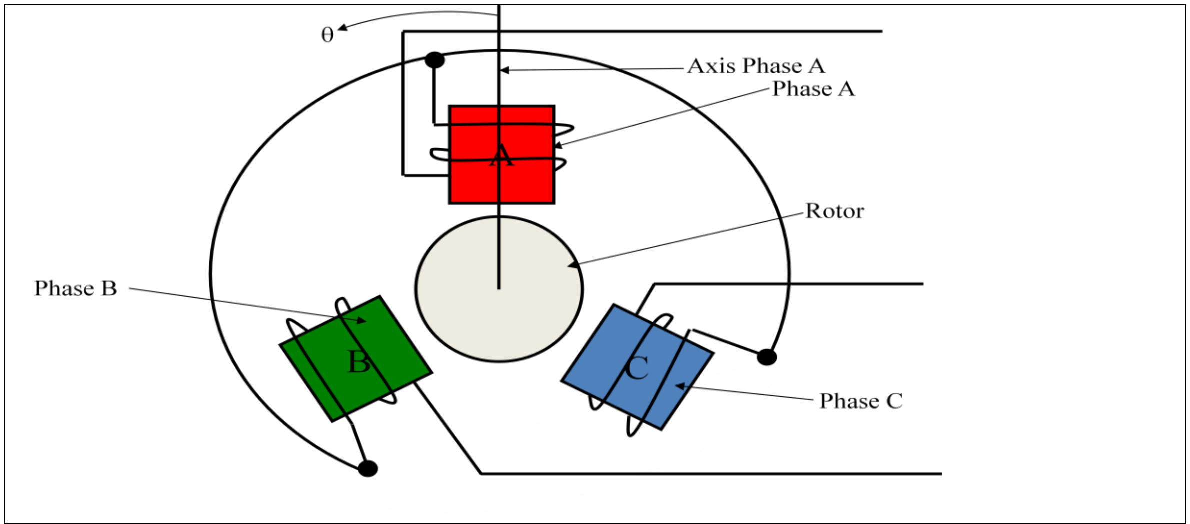

Figure 1: Cross section of a simple induction motor

Principles of Operation of Induction Motor (Graphical Treatment)

Let the synchronous frequency ω be 1rad/sec. Hence, the spatial distribution of resultant flux at t=0sec, t=60sec, t=120sec, t=180 sec, t=240sec and t=300sec and are shown in **Figure 2**. The explanation of the flux creation is as follows

- At t=0, phase **A** is a maximum north pole, while phase **B** and phase **C** are weak south poles, **Figure (2a)**.
- At t=60, phase **C** is a strong south pole, while phase **B** and phase **A** are weak north poles **Figure (2b)**.
- At t=120, phase **B** is a strong north pole, while phase **A** and phase **C** are weak south poles **Figure (2c)**.
- At t=180, phase **A** is a strong south pole, while phase **B** and phase **C** are weak north poles **Figure (2a)**.
- At t=240, phase **C** is a strong north pole, while phase **A** and phase **B** are weak south poles **Figure (2e)**.
- At t=300, phase **B** is s strong south pole, while phase **C** and phase **A** are weak north poles **Figure (2f)**.

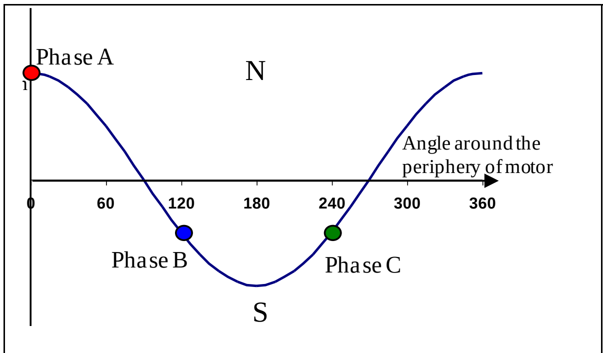

Figure 2a: Magnetic poles position at t=0

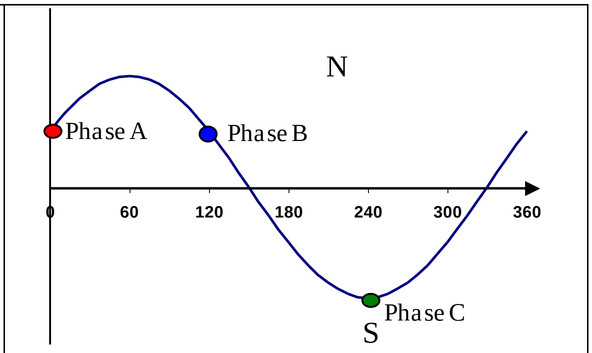

Figure 2b: Magnetic poles position at t=60

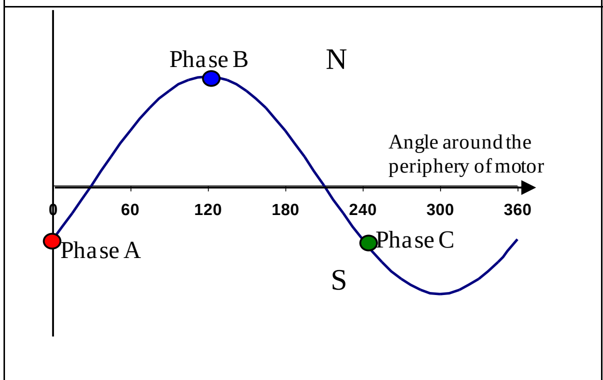

Figure 2c: Magnetic poles position at t=120

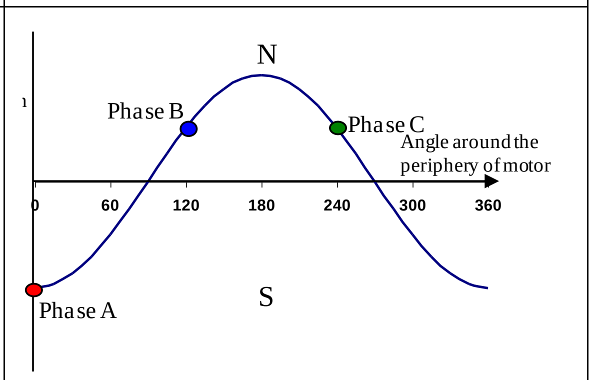

Figure 2d: Magnetic poles position at t=180

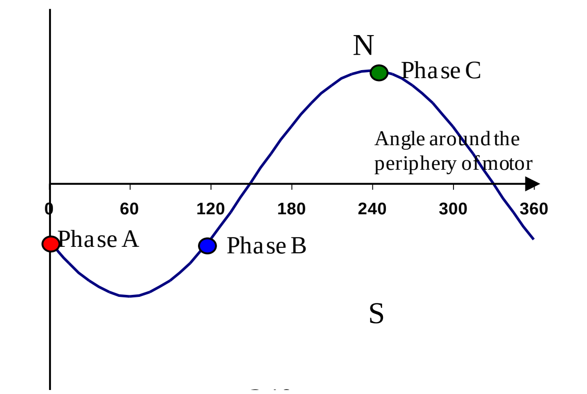

Figure 2e: Magnetic poles position at t=240

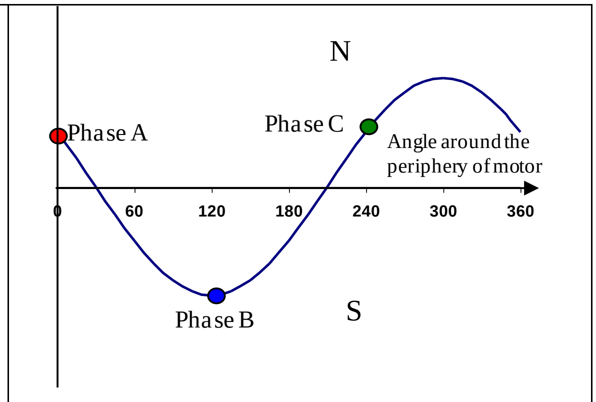

Figure 2f: Magnetic poles position at t=300

Fluxes and MMF in Induction Motor

Although the flux generated by each coil is only alternating flux, the combined flux contributions of the three coils, carrying current at appropriate sequential phase angles, produces a two pole rotating flux. The rotating flux produced by three phase currents in the stationary coils, may be linked to the rotating field produced by a magnet sweeping around the rotor (**Figure 3a**). The rotating field cuts the rotor bars in its anti clockwise sweep around the rotor. According to Lenz's law, the voltage, current and flux generated by the relative motion between a conductor and a magnetic field will be in a direction to oppose the relative motion. From **Figure 3a** it can be seen that the bars **a** and **b** are just under the pole centers and have maximum electromotive force (e.m.f) generated in them and this is indicated by large cross and dots. The bars away from the pole centers have reduced magnitude of generated e.m.fs and these are indicated by varying sizes of dots and crosses. If the rotor circuit is assumed purely resistive, then current in any bar is in phase with the e.m.f generated in that bar (**Figure 3a**). The existence of currents in the rotor circuit gives rise to rotor mmf F_2, which lags behind airgap flux ϕ_m by a space angle of 90°. The rotor mmf causes the appearance two poles N_2 and S_2.The relative speed between the poles N_1, S_1 and the rotor poles N_2, S_2 is zero. Rotating pole N_1 repels N_2 but attracts S_2. Consequently the electromagnetic torque developed by the interaction of the airgap flux ϕ_m and the rotor mmf F_2 is in the same direction as that of the rotating magnetic field (**Figure 3b**). The space phase angle between F_2 and ϕ_m is called the load angle and for this case it is 90° (**Figure 3b**). The torque produce is given by

$$T_e = k\phi F_2 \sin(\frac{\pi}{2}) = k\phi F_2 \qquad (5)$$

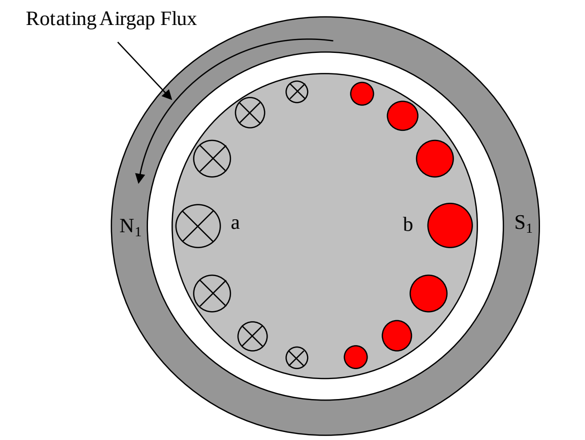

Figure 3a: Rotating airgap flux

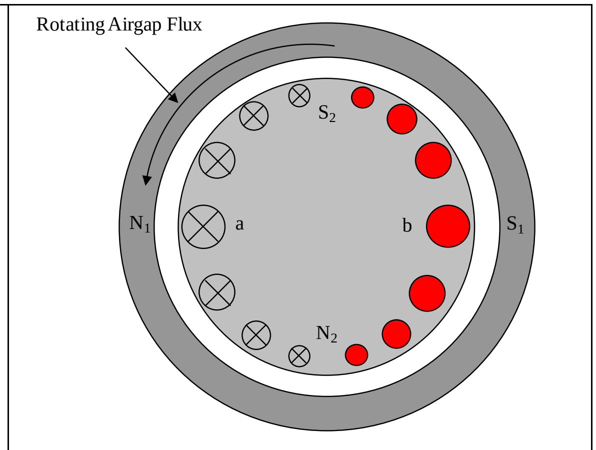

Figure 3b: Rotor and Stator fields

In actual machine, the rotor bars are embedded in the iron, hence the rotor circuit has leakage reactance. Thus the rotor current in each bar lags behind the generated e.m.f in that bar by rotor power factor angle:

$$\theta_2 = \tan^{-1}\frac{x_2}{r_2} \tag{6}$$

From **Figure 4** it is seen that bars **a** and **b** under the poles have a maximum generated e.m.fs. On account of the rotor reactance (x_2), the currents in these bars will be maximum only when the poles N_1, S_1 have traveled through an angle θ_2 (**Figure 4**). The rotor current generates rotor mmf F_2 is space displaced from the air gap flux ϕ_m by a load angle $\theta_2 + \frac{\pi}{2}$.

The torque produced by the motor in this situation is

$$T_e = k\phi F_2 \sin(\frac{\pi}{2} + \theta_2) \tag{7}$$

Greater the value of x_2, greater is the departure of load angle from its optimal value of $\frac{\pi}{2}$ and lesser is the torque. To generate a high starting torque, θ_2 should be made as small as possible and this is done by increasing rotor resistance r_2.

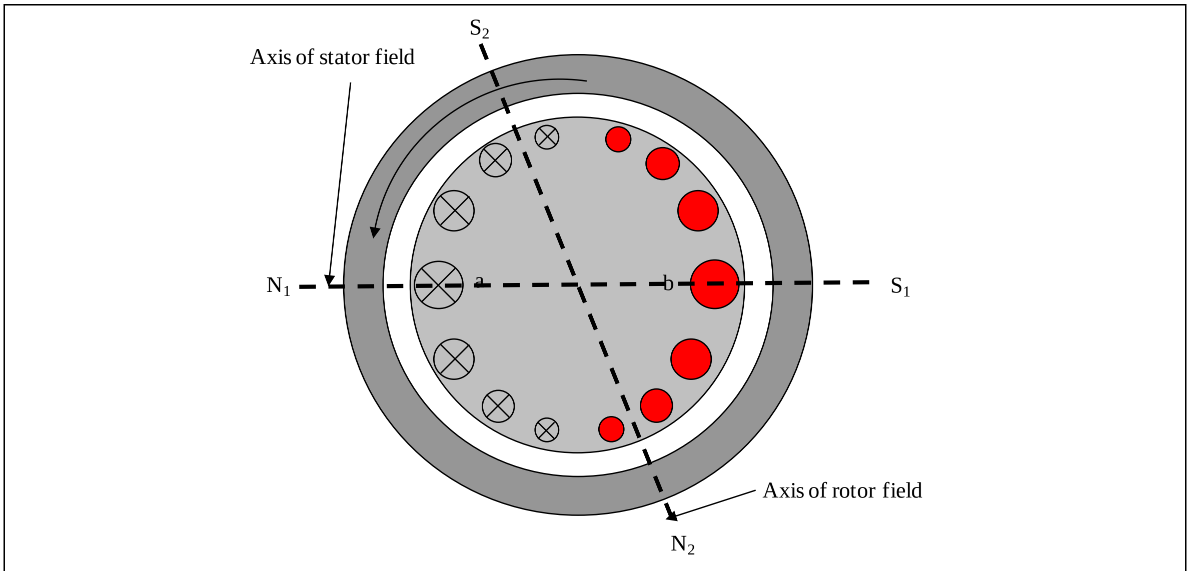

Figure 4: Axis of rotor and stator fields in case of rotor with inductance

Rotor Action

At standstill, rotor conductors are being cut by rotating flux wave at synchronous speed n_s. Hence, the frequency f_2 of the rotor e.m.f and current is equal to the input voltage frequency f_1. When the rotor rotates at a speed of n_r rotations per second (r.p.s) in the direction of rotating flux wave, the relative speed between synchronously rotating flux and rotor conductors becomes $(n_s - n_r)$ r.p.s, i.e.,

$$f_2 = \frac{P(n_s - n_r)}{2} \tag{8}$$

where P is the number of poles of the machine

Hence, the slip of the machine is defined as

$$s = \frac{n_s - n_r}{n_s} \tag{9}$$

Thus, the rotor frequency is defined as

$$f_2 = \frac{P \times s \times n_s}{2} = sf_1 \tag{10}$$

At standstill the rotor frequency is f_1 and the field produced by rotor currents revolves at a speed equal to $2f_1/P$ w.r.t. rotor structure. When the rotor rotates at a speed of nr, the rotor frequency is sf_1 and the rotor produced field revolves at a speed of $2(sf_1)/P = sn_s$ w.r.t. rotor structure. The rotor is already rotating at a speed of n_r w.r.t. stator. Hence, the speed of rotor field w.r.t. to stator is equal to the sum of mechanical rotor speed nr and rotor field speed sn_s w.r.t. rotor. Hence, the speed of the rotor field with respect to stator is given by

$$n_r + sn_s = n_s(1-s) + sn_s = n_s \text{ r.p.s} \tag{11}$$

The stator and rotor fields are stationary with respect to each other at all possible rotor speeds. Hence, a steady torque is produced by their interaction. The rotor of an induction motor can never attain synchronous speed. If does so then the rotor conductors will be stationary w.r.t. the synchronously rotating rotor conductors and hence, rotor m.m.f. would be zero.

Rotor e.m.f and Equivalent Circuit

Let the rotor e.m.f. at standstill be E_2. When the rotor speed is $0.4n_s$, the slip is 0.6 and the relative speed between rotating field and rotor conductors is $0.6n_s$. Hence, the induced e.m.f. , per phase, in the rotor is

$$0.6n_s \frac{E_2}{n_s} = 0.6E_2 \tag{12}$$

In general, for any value of slip s, the per phase induced e.m.f in the rotor conductors is equal to sE_2. The other quantities of the rotor are given as

The rotor leakage reactance at standstill is $x_2 = 2\pi f_1 L_2$ (13a)

The rotor leakage reactance at any slip s is $2\pi s f_1 L_2 = s x_2 \Omega$ (13b)

The rotor leakage impedance at standstill is $\sqrt{r_2^2 + x_2^2}$ (13c)

At any slip s rotor leakage impedance is $\sqrt{r_2^2 + (sx_2)^2}$ (13d)

The per phase rotor current at standstill is $\dfrac{E_2}{\sqrt{r_2^2 + x_2^2}}$ (13e)

The per phase rotor current at any slip s is $\dfrac{sE_2}{\sqrt{r_2^2 + (sx_2)^2}} = \dfrac{E_2}{\sqrt{\left(\frac{r_2}{s}\right)^2 + (x_2)^2}}$ (13f)

Based on **equation 13f** the equivalent circuit of the rotor is shown in **Figure 5**.

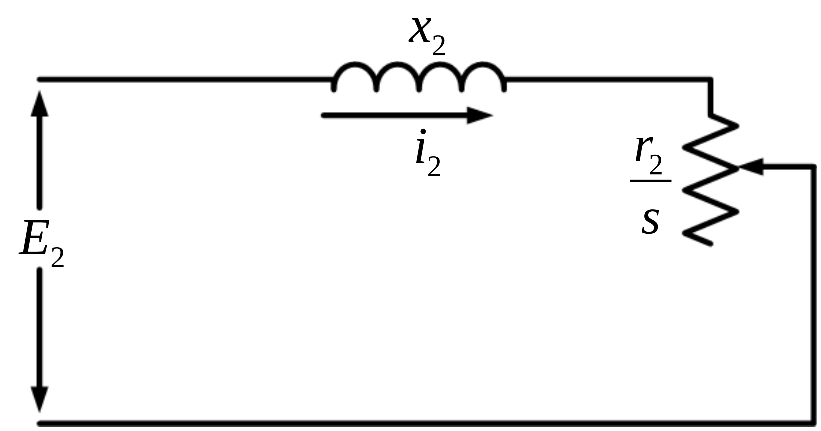

Figure 5: Equivalent circuit of rotor

Complete Equivalent Circuit

The rotating air gap flux generates back e.m.f. (E_1) in all the three phases of the stator. The stator applied terminal voltage V_1 has to overcome back e.m.f. E_1 and the stator leakage impedance drop:

$$V_1 = E_1 + I_1(r_1 + jx_1) \tag{14}$$

The stator current I_1 consists of following two components, I_1' and I_m. The component I_1' is the load component and counteracts the rotor m.m.f. The other component I_m creates the resultant air gap flux ϕ_m and provides the core loss. This current can be resolved into two components: I_c in phase with E_1 and I_ϕ lagging E_1 by 90°. In the equivalent circuit of the stator shown in **Figure 6**, I_c and I_ϕ are taken into account by a parallel branch, consisting of core-loss resistance R_c in parallel to magnetizing reactance X_ϕ.

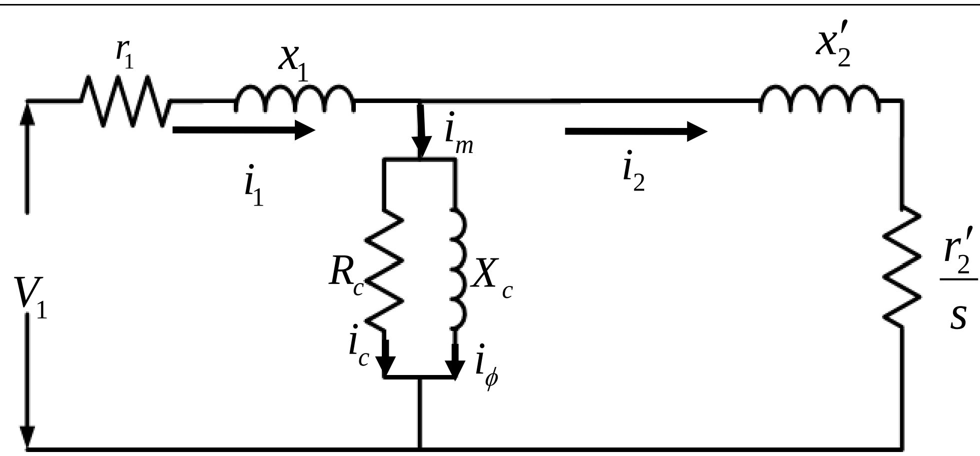

Figure 6: Equivalent circuit of stator

The rotor e.m.f. E_2 when referred to stator becomes

$$E_1 = \frac{E_2}{N_2^{'}} N_1^{'} \tag{15}$$

where $N_1^{'}$ and $N_2^{'}$ are number of turns in the stator and rotor respectively

The rotor leakage impedance when referred to the stator is

$$Z_2' = \left(\frac{r_2}{s} + jx_2\right)\left(\frac{N_1^{'}}{N_2^{'}}\right)^2 = \frac{r_2^{'}}{s} + jx_2^{'}$$

$$\textit{where} \tag{16}$$

$$r_2' = \left(\frac{N_1'}{N_2'}\right)^2 r_2;\; x_2 = \left(\frac{N_1'}{N_2'}\right)^2 x_2$$

After referred the rotor quantities towards stator, the combined equivalent circuit of the machine is shown in **Figure 7**. For simplicity the prime notations will not be used in the further discussions and all the rotor quantities henceforth will be referred to the stator side. Moreover, all the quantities are at stator frequency.

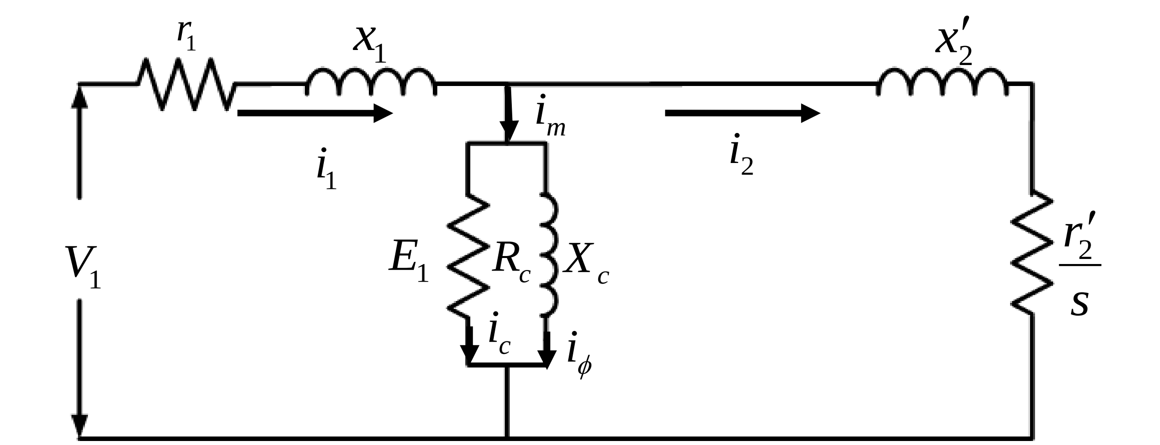

Figure 7: Complete equivalent circuit of Induction Motor

Simplification Equivalent Circuit

The use of exact equivalent circuit is laborious; hence some simplifications are done in the equivalent circuit. Under normal operating conditions of constant voltage and frequency, core loss in induction motors is usually constant. Hence, the core loss component can be omitted from the equivalent circuit, **Figure 8**. However, to determine the shaft power, the constant core loss must be taken into account along with friction, windage and stray load losses. It should be noted that all the quantities used in the equivalent circuit are per phase quantities. Steady state performance parameters of the induction motor, such as current, speed, torque, losses etc. can be computed from the equivalent circuit shown in **Figure 8**.

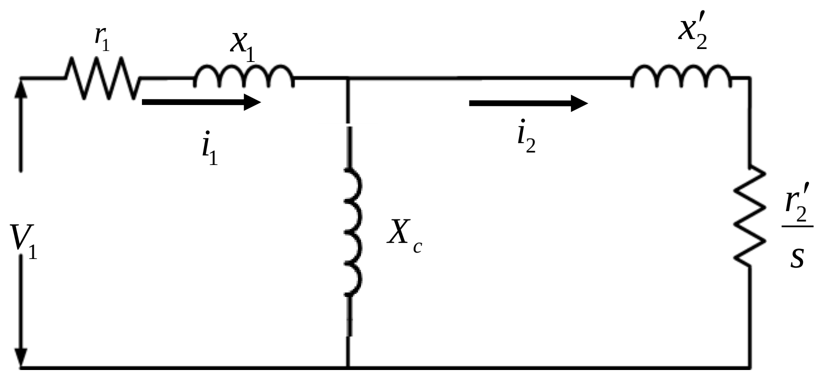

Figure 8: Simplified equivalent circuit of Induction Motor

Analysis of Equivalent Circuit

The total power transferred across the air gap (P_g) from the stator is

$$P_{gap} = n_{ph} i_2^2 \left(\frac{r_2}{s}\right)$$
$$= n_{ph}\left(i_2^2 r_2 + i_2^2 r_2\left(\frac{1-s}{s}\right)\right) \tag{17}$$

where n_{ph} is the number of phases

Hence, the rotor Ohmic loses and the internal mechanical power are given as

$$P_{rotor} = n_{ph} i_2^2 r_2 = n_{ph} s P_{gap} \tag{18}$$

$$P_{mech} = n_{ph} i_2^2 r_2 \left(\frac{1-s}{s}\right) \tag{19}$$

The internal (gross or air gap or the electromagnetic torque) torque developed per phase is given by

$$T_e = \frac{P_{mech}}{\omega_r} = \frac{(1-s)P_{gap}}{(1-s)\omega_s} = \frac{P_{gap}}{\omega_s} \tag{20}$$

where ω_r is the rotor speed and ω_s is the synchronous speed

The output or the shaft power is

$$P_{shaft} = P_{mech} - Mechanical\ losses$$
$$or \tag{21}$$
$$P_{shaft} = P_{gap} - Rotor\ Ohmic\ losses - Mechanical\ losses$$

Thevenin's Equivalent Circuit of Induction Motor

When the torque-slip or power-slip characteristics are required, application of Thevenin's theorem to the induction motor equivalent circuit reduces the computation complexity. For applying Thevenin's theorem to the equivalent circuit shown in **Figure 8**, two points ***a***, ***b*** are considered as shown in **Figure 9**. From these points the voltage source V_1 is viewed and the equivalent voltage at point ***a*** and ***b*** is

$$V_{eq} = \frac{V_1\left(jX_c\right)}{R_1 + j\left(X_1 + X_c\right)} \tag{22}$$

The equivalent impedance of the circuit as seen from points ***a*** and ***b*** is

$$Z_{eq} = \frac{\left(R_1 + jX_1\right)\left(jX_c\right)}{R_1 + j\left(X_1 + X_c\right)} \tag{23}$$

For most induction motors ($X_1 + X_c$) is much greater than R_1. Hence, R_1 can be neglected from the denominator of **equation 22** and **equation 23**. The simplified expression for V_{eq} and Z_{eq} are

$$V_{eq} = \frac{V_1\left(jX_c\right)}{j\left(X_1 + X_c\right)} = \frac{V_1 X_c}{X_1 + X_c} \tag{24}$$

$$Z_{eq} = R_{eq} + jX_{eq} = \frac{R_1 X_c}{X_1 + X_c} + \frac{jX_1 X_c}{X_1 + X_c} \tag{25}$$

From the Thevenin's equivalent circuit, the rotor current can be determined as

$$I_2 = \frac{V_{eq}}{\left(R_{eq} + \frac{r_2}{s}\right) + j\left(X_{eq} + X_2\right)} \tag{26}$$

The airgap torque produced by the motor is

$$T_e = \frac{n_{ph}}{\omega_s} \frac{V_{eq}^2}{\left(R_{eq} + \frac{r_2}{s}\right)^2 + \left(X_{eq} + X_2\right)^2} \frac{r_2}{s}$$
$$= \frac{K_t}{\left(R_{eq} + \frac{r_2}{s}\right)^2 + X^2} \frac{r_2}{s} \qquad (27)$$

where

$$K_t = \frac{n_{ph} V_{eq}^2}{\omega_s} \text{ and } X = X_2 + X_{eq}$$

A typical torque versus slip curve for IM obtained from **equation 27** is shown in **Figure 10**.

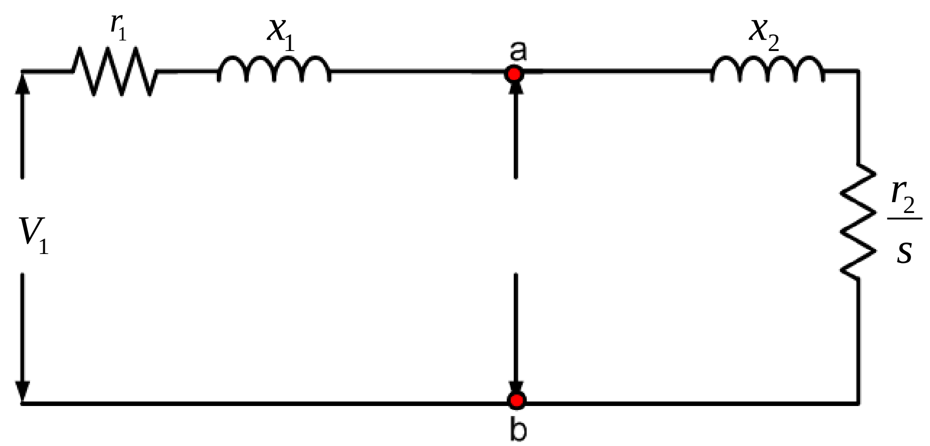

Figure 9: Thevenin's equivalent circuit

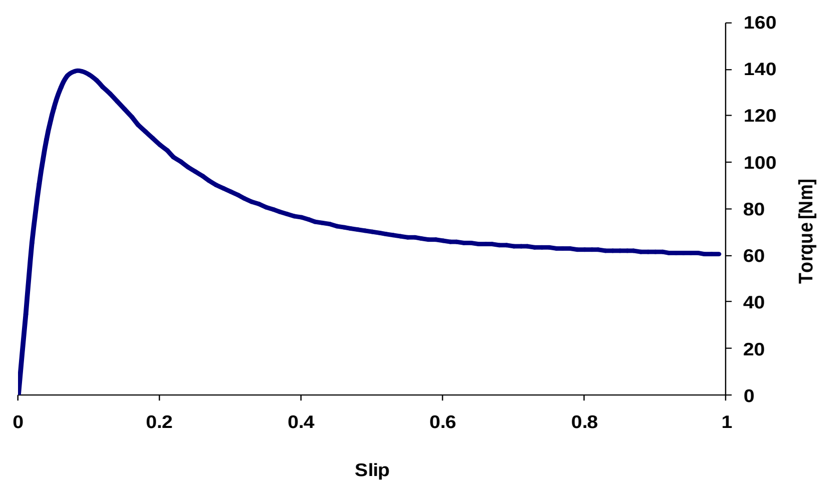

Figure 10: Torque vs. Slip curve of IM

Suggested Reading:

[1] M. G. Say, *The Performance and Design of Alternating Current Machines*, CBS Publishers, New Delhi

[2] S. J. Chapman, *Electric Machinery Fundamentals*, McGraw Hill, 2005

Lecture 19: Permanent magnet motors, their configurations and optimization

Control of Induction Motors

Introduction

The topics covered in this chapter are as follows:

- Speed Control of Induction Motor
- Constant Volts/Hz control
- Implementation of Constant Volts/Hz Control
- Steady State Analysis of IM with Constant Volts/Hz Control

Speed Control of Induction Motor (IM)

Speed control of IM is achieved in the inverter driven IM by means of variable frequency. Besides the frequency, the applied voltage needs to be varied to keep the air gap flux constant. The induced e.m.f in the stator winding of an ac machine is given by

$E_1 = 4.44 k_{w1} \phi_m f_s N_1$

where

k_{w1} is the stator winding factor

ϕ_m is the peak airgap flux (1)

f_s is the supply frequency

N_1 is the number of turns per phase in the stator

The stator applied terminal voltage V_1 (**Figure** 1) has to overcome back e.m.f. E_1 and the stator leakage impedance drop (refer Lecture 17):

$$V_1 = E_1 + i_1(r_1 + jx_1) \quad (2)$$

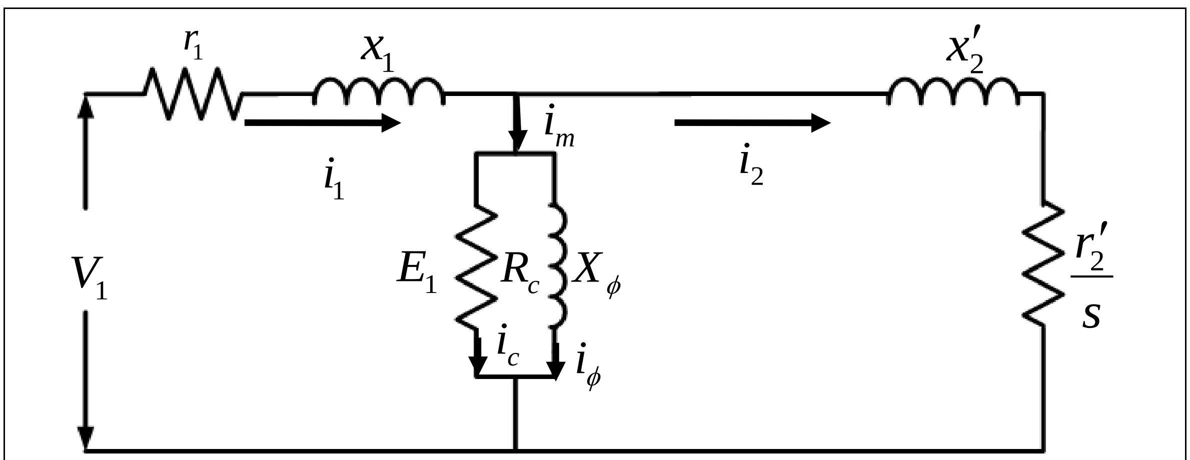

Figure 1: Equivalent circuit of IM

If the stator impedance ($r_1 + jx_1$) is neglected, the induced e.m.f approximately equals the supply phase voltage. Hence,

$$V_1 \cong E_1 \tag{3}$$

Substituting for E_1 from **equation 1** into **equation 2** gives the flux as

$$\phi_m \cong \frac{V_1}{K_b f_s}$$

where (4)

$k_b = 4.44 k_{w1} N_1$ is the flux constant

Since the factor k_b is constant, from **equation 4** it can be seen that ***the flux is proportional to the ratio between the supply voltage and frequency***. Hence,

$$\phi_m \propto \frac{V_1}{f_s} \propto k_{vf} \tag{5}$$

where k_{vf} is the ration bewteen V_1 and f_s

From **equation 5**, it is seen that, to maintain the flux constant k_{vf} has to be maintained constant. Hence, whenever the stator frequency (f_s) is changed for speed control, the stator input voltage (V_1) has to be changed accordingly to maintain the airgap flux (ϕ_m) constant. A number of control strategies have been developed depending on how the voltage to frequency ratio is implemented:

- Constant volts/Hz control
- Constant slip-speed control
- Constant air gap control
- Vector Control

The constant volts/Hz strategy is explained in this lecture.

Constant Volts/Hz Control

The **equation 2** is converted into per unit (p.u) as

$$\frac{V_1}{V_b} = \frac{E_1 + I_1(r_1 + jx_1)}{V_b}$$

$$\Rightarrow V_{1n} = E_{1n} + I_{1n}\left(r_{1n} + jx_{1n}\right)$$

where

$$V_{1n} = \frac{V_1}{V_b};\ I_{1n} = \frac{I_{1n}}{I_b};\ r_{1n} = \frac{I_b r_1}{V_b}$$

$$E_{1n} = \frac{E_1}{V_b} = \frac{jX_\phi i_\phi}{V_b} = \frac{j\omega_s L_\phi i_\phi}{V_b} = \frac{j\omega_s L_\phi i_\phi}{\lambda_b \omega_b} = j\left(\frac{\lambda_\phi}{\lambda_b}\right)\left(\frac{\omega_s}{\omega_b}\right) \tag{6}$$

$$x_{1n} = \frac{I_b x_1}{V_b} = \frac{I_b \omega_s L_{1s}}{V_b} = L_{1n}\omega_{sn}$$

$$V_b = \lambda_b \omega_b$$

V_b is the base voltage

I_b is the base current

λ_b is the flux linkage (flux linkage is rate of change of flux with repect to time)

Hence, **equation 2** in p.u form is written as

$$V_{1n} = E_{1n} + i_{1n}(r_{1n} + jx_{1n}) = i_{1n}r_{1n} + j\omega_{sn}\left(L_{1n}i_{1n} + \lambda_{\phi n}\right)$$

where (7)

$$E_{1n} = jL_{\phi n}\omega_{sn}i_{\phi n} = \lambda_{\phi n}\omega_{sn}$$

$$\lambda_{\phi n} = jL_{\phi n}i_{\phi n}$$

The magnitude of the input voltage ($|V_1|$) is given as

$$\left|V_{1n}\right| = \sqrt{\left(i_{1n}r_{1n}\right)^2 + \omega_{sn}^2\left(L_{1n}i_{1n} + \lambda_{\phi n}\right)^2} \tag{8}$$

For a constant air gap flux linkages of 1pu, the pu applied voltage vs. p.u stator frequency is shown in **Figure 2**. The values of r_{1n} and x_{1n} used to obtain the plot of **Figure 2** are 0.03 and 0.05 pu respectively.

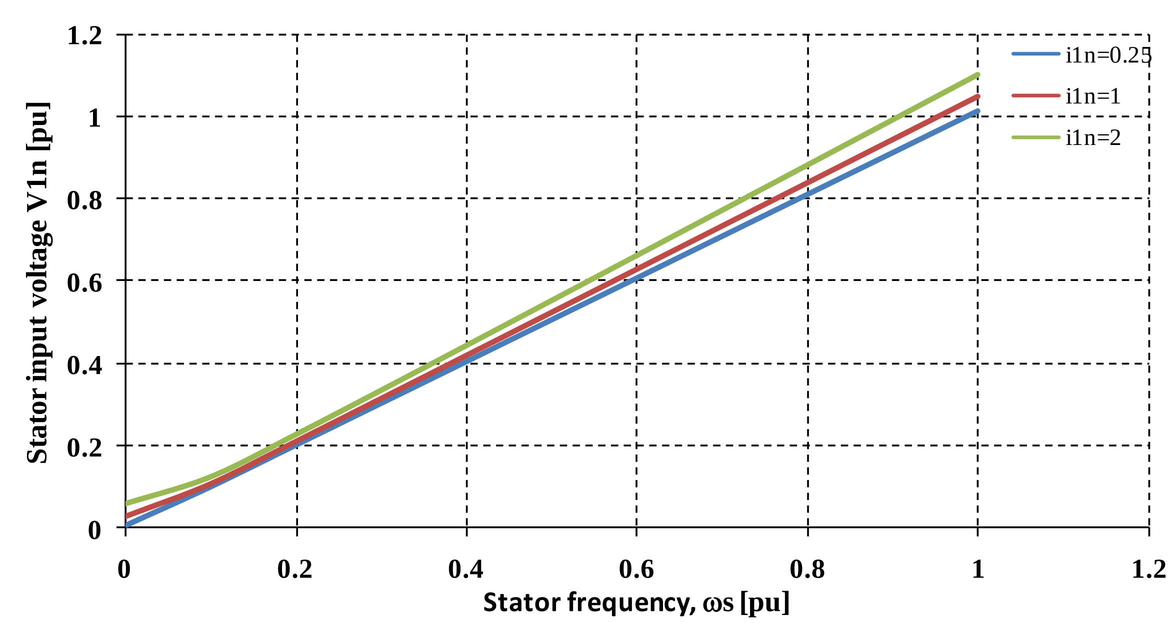

Figure 2: PU stator phase voltage vs. pu stator frequency

From **equation 8** it can be seen that the volts/Hz ratio needs to be adjusted in dependence on the frequency, the air gap flux magnitude, the stator impedance and the magnitude of the stator current. The relationship between the applied phase voltage and the frequency is written as

$$V_{1n} = V_{on} + k_{vf} f_{sn} \tag{9}$$

From **equation 8** the parameters V_o and k_{vf} is obtained as

$$\begin{aligned} V_{on} &= i_{1n} r_{1n} \\ k_{vf} &= \omega_{sn}\left(\lambda_{\phi n} + L_{1n} i_{1n}\right) \end{aligned} \tag{10}$$

The parameter V_o is the offset voltage required to overcome the stator resistive drop. In case the IM is fed by a DC-AC converter, the fundamental r.m.s phase voltage for 180^o conduction is given by (refer **Lecture 15**):

$$V_1 = \frac{v_{as}}{\sqrt{2}} = \frac{2}{\pi}\frac{V_{dc}}{\sqrt{2}} = 0.45V_{dc} \tag{11}$$

where

V_{dc} is the input dc voltage to DC-AC converter

The **equation 11** can be written in pu form as

$$V_{1n} = \frac{V_1}{V_b} = 0.45V_{dcn} \tag{12}$$

Substituting the value of V_{1n} into **equation 9** gives

$$0.45V_{dcn} = V_{on} + k_{vf}\, f_{sn}$$

Implementation of Constant Volts/Hz Control

The implementation of volts/Hz strategy is shown in **Figure 3**.

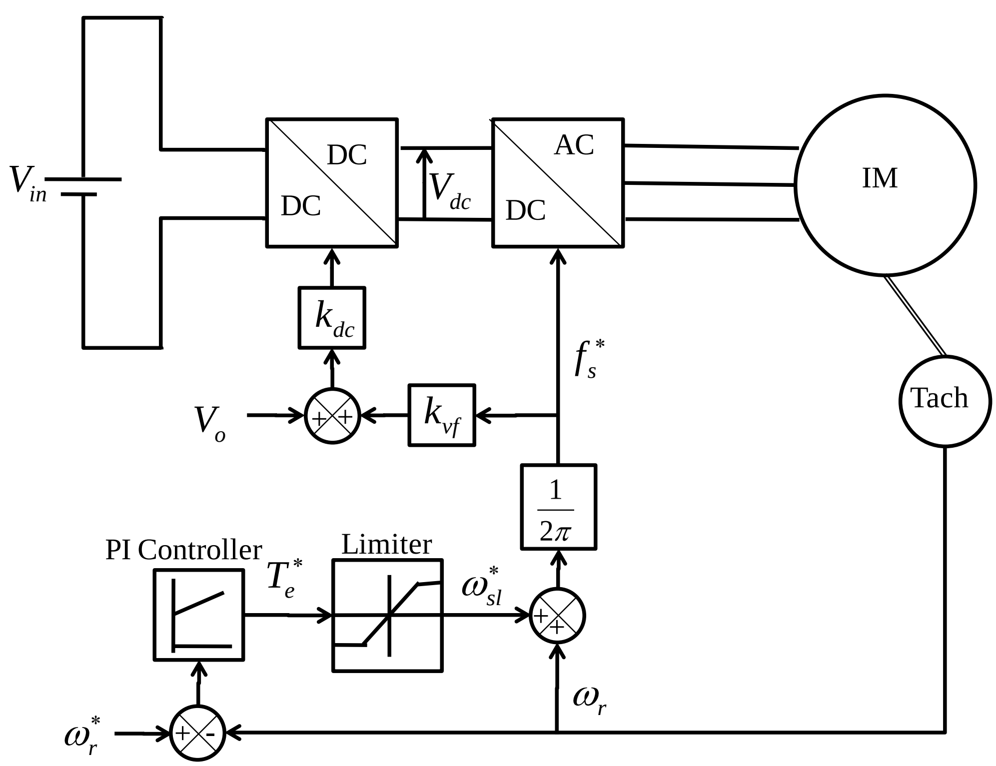

Figure 3: Closed loop induction motor drive with constant volts/Hz control strategy [1]

The working of the closed loop control shown in **Figure 3** is as follows:

a. The actual rotor speed (ω_r) is compared with its desired value ω_r^* and the error is passed through a PI controller.
b. The output of the PI controller is processed through a limiter to obtain the slip-speed command ω_{sl}^*. The limiter ensures that ω_{sl}^* is within the maximum allowable slip speed of the induction motor.
c. The slip speed command is added to electrical rotor speed ω_r to obtain the stator frequency command f_s^*.
d. The frequency command f_s^* is enforced in the inverter and the corresponding dc link voltage (V_{dc}) is controlled through the DC-DC converter.
e. The offset voltage V_1^* is added to the voltage proportional to the frequency and multiplied by k_{dc} to obtain the dc link voltage.

Steady State Performance of IM with Constant Volts/Hz Control

The steady state performance of the constant-volts/Hz controlled induction motor is computed by using the applied voltage given in **equation 9**. Using the equivalent circuit of IM, the following steps are taken to compute the steady state performance:

a. Start with a minimum stator frequency and a very small slip
b. Compute the magnetization, core-loss, rotor and stator phase current
c. Calculate the electromagnetic torque, power, copper and core losses
d. Calculate the input power factor and efficiency.
e. Increase the slip and go to ***step b*** unless maximum desired slip is reached.
f. Increase the stator frequency and go to ***step a*** unless maximum desired frequency is reached.

In **Figure 4** the characteristics of volts/Hz control of an IM is shown. The parameters of the IM are as follows:

Applied stator line to line voltage $V_{ll} = 200V$

Frequency of applied voltage $f_s = 50Hz$

Rated Output Power $P_{out} = 3kW$

Stator resistance $r_1 = 0.3\Omega$

Stator leakage inductance $L_1 = 0.001H$

Rotor resistance $r_2 = 0.2\Omega$

Rotor leakage inductance $L_2 = 0.0015H$

Efficiency $\eta = 0.8$

Power factor $pf = 0.85$

Connection of phases: Y

Based on the above parameters of the motor, the base quantities are determined as follows:

Base speed $\omega_{base} = 2\pi f_S = 2\times\pi\times 50 = 314.16\ rad/s$

Base voltage $V_{base} = V_{ph} = \dfrac{200}{\sqrt{3}} = 115.47\ V$

Base power $P_{base} = \text{Rated Output Power} = 3000\ \text{W}$

Base current $I_{base} = \dfrac{P_{base}}{3\times V_{base}\times\eta\times pf\times V_{base}} = 12.74\ A$

Base Torque $T_{base} = \dfrac{P_{base}}{\omega_{base}} = \dfrac{3000}{314.16} = 9.55\ Nm$

After having calculated the base values, the torque produced by the IM is calculated using the following expression (**equation 27** of Lecture 17):

$$T_e = \frac{n_{ph}}{\omega_s}\frac{V_{eq}^2}{\left(R_{eq}+\dfrac{r_2}{s}\right)^2+\left(X_{eq}+X_2\right)^2}\frac{r_2}{s} \tag{13}$$

The pu torque T_{en} is given by

$$T_{en} = \frac{T_e}{T_{base}} \tag{14}$$

In order to obtain the curve shown in **Figure 4**, the torque is calculated for different values of slip and frequency as described algorithm above. Using the constant volts/Hz control, the IM can be operated up to rated frequency. However, if it is required to operate the motor beyond rated speed then ***Flux weakening operation*** is used.

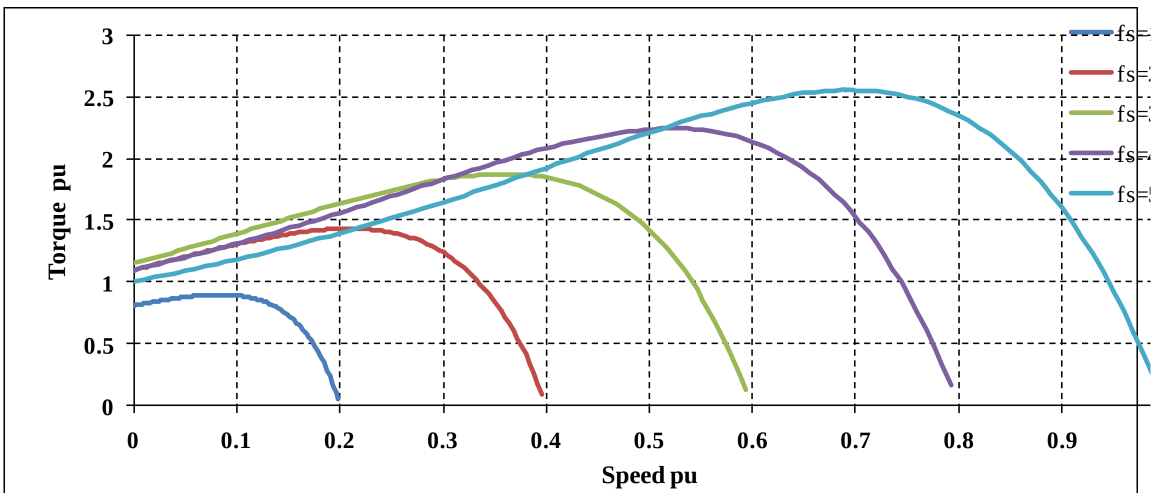

Figure 4: Torque vs. Speed Curve for Constant volts/Hz control of IM

From **Figure 4** the following points can be observed:

- As the frequency of the stator input voltage increases, the maximum speed of the motor increases.
- With increase in frequency, the maximum torque produced by the motor also increases.
- The starting torque (torque at zero speed) does not vary much with increase in frequency.

In **Figure 5** the Torque vs. Speed curve for higher offset voltage $V_o = 7 i_{1n} r_{1n}$ is shown. It can be seen that using a higher offset voltage:

- the starting torque has increased.
- the maximum torque produced by the motor at different frequencies is almost constant.

Here the motor is operated at 10Hz between the rotor speeds ω_{r1} and ω_{r2}, at 20 Hz between ω_{r2} and ω_{r3} and so on. With this operation constant torque is maintained almost up to rated speed.

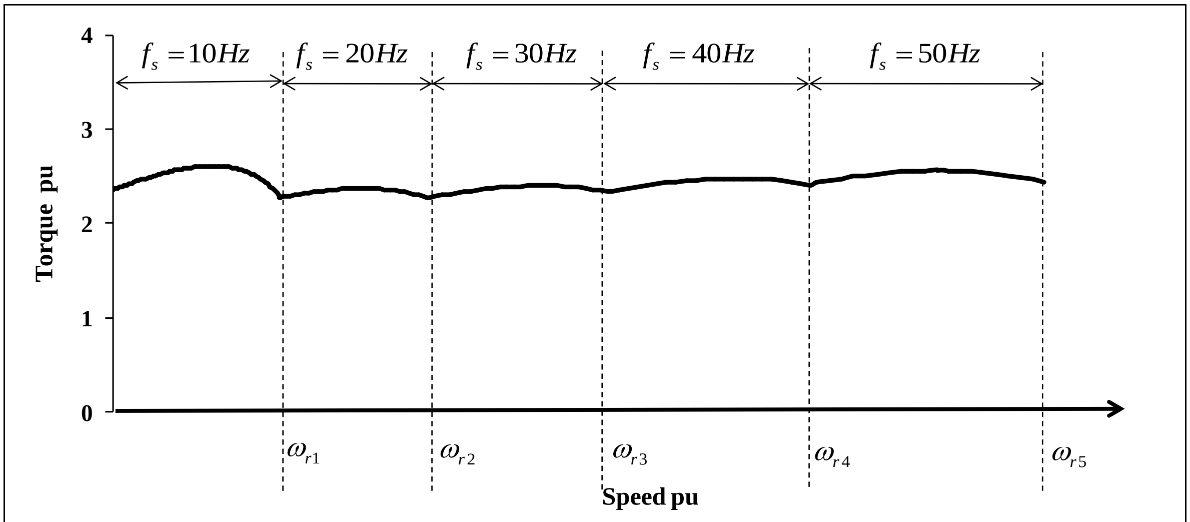

Figure 5: Constant Torque vs. Speed Curve for Constant volts/Hz control of IM

Utilizing the second point, a constant torque can be obtained from starting condition up to rated speed as shown in **Figure 6**.

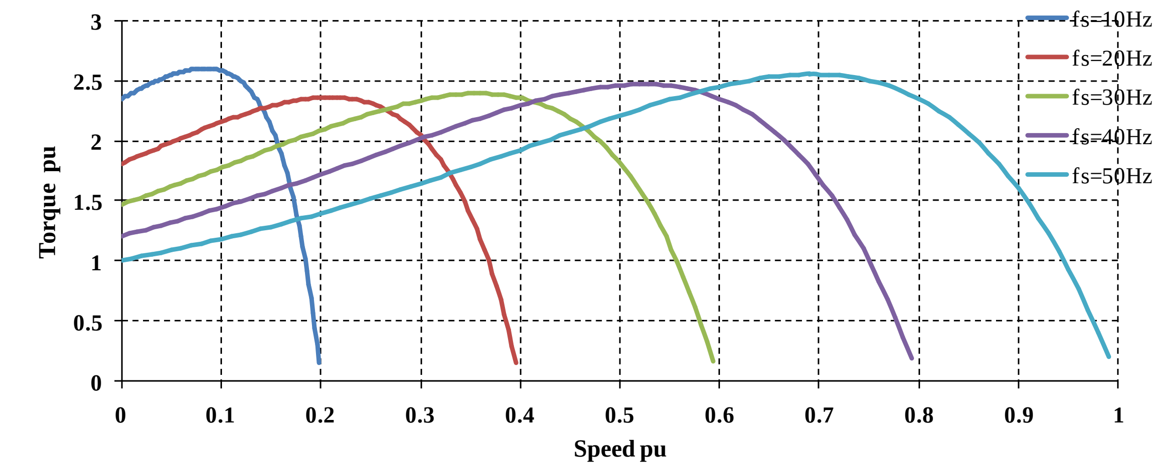

Figure 6: Torque vs. Speed Curve for Constant volts/Hz control of IM at higher offset voltage

The power factor vs. slip, stator current vs. slip curves and torque vs. slip for constant volts/Hz control are shown in **Figure 7**, **8** and **9** respectively. From **Figures 7-9** the following can be observed:

- As the slip increases (speed decreases) the power factor of the motor decreases. It attains a maximum value at a small slip (s_{pf}) value and then drops sharply.
- As the frequency increases the slop of the power factor between s_{pf} and unity slip increases.
- For any given slip, the magnitude of the stator current increases as the frequency increases. The magnitude of torque at a given slip also increases with increase in slip with the exception of unity slip (starting condition).

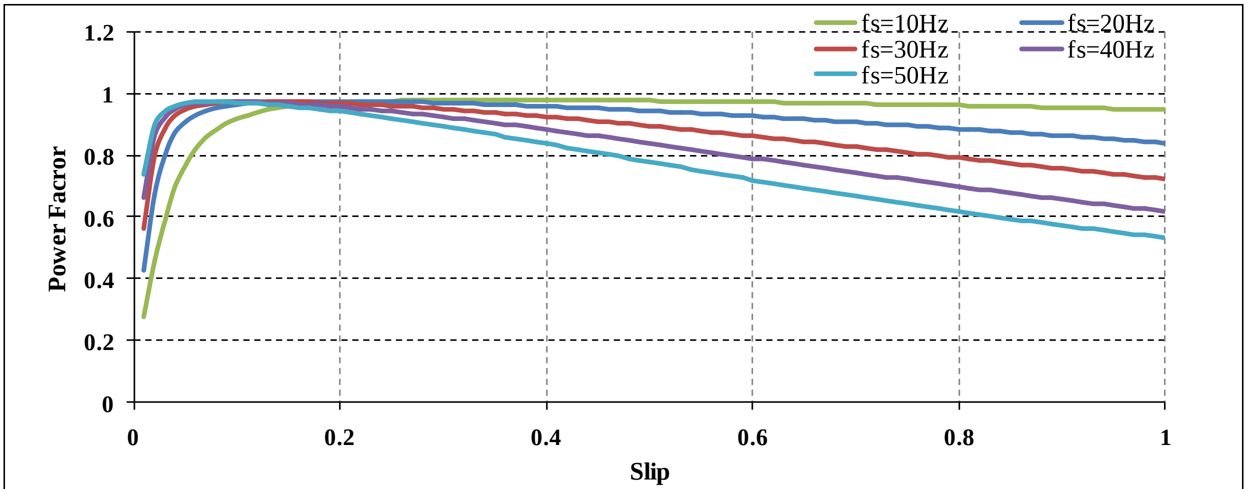

Figure 7: Power factor vs. Slip Curve for Constant volts/Hz control of IM

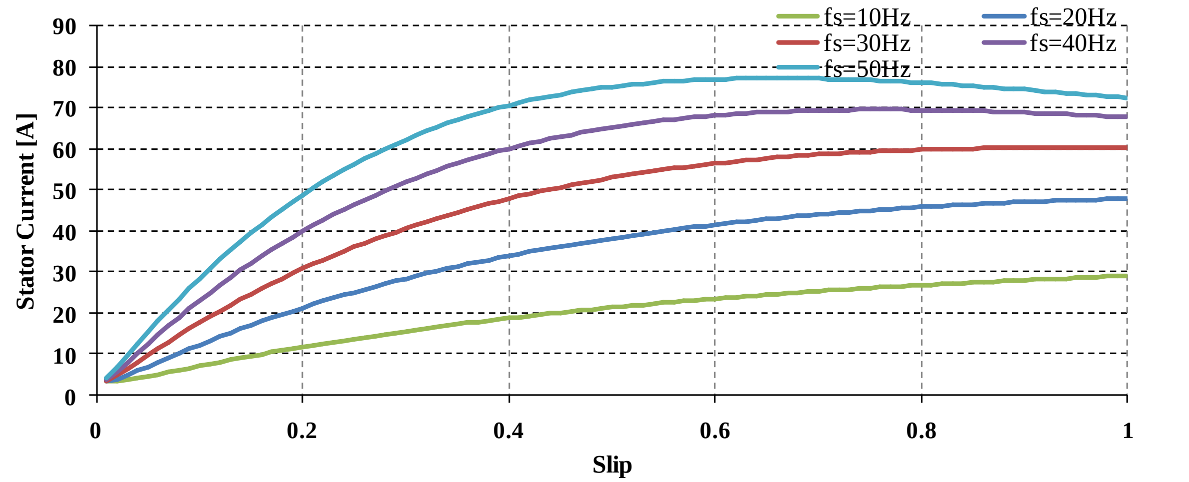

Figure 8: Stator Current vs. Slip Curve for Constant volts/Hz control of IM

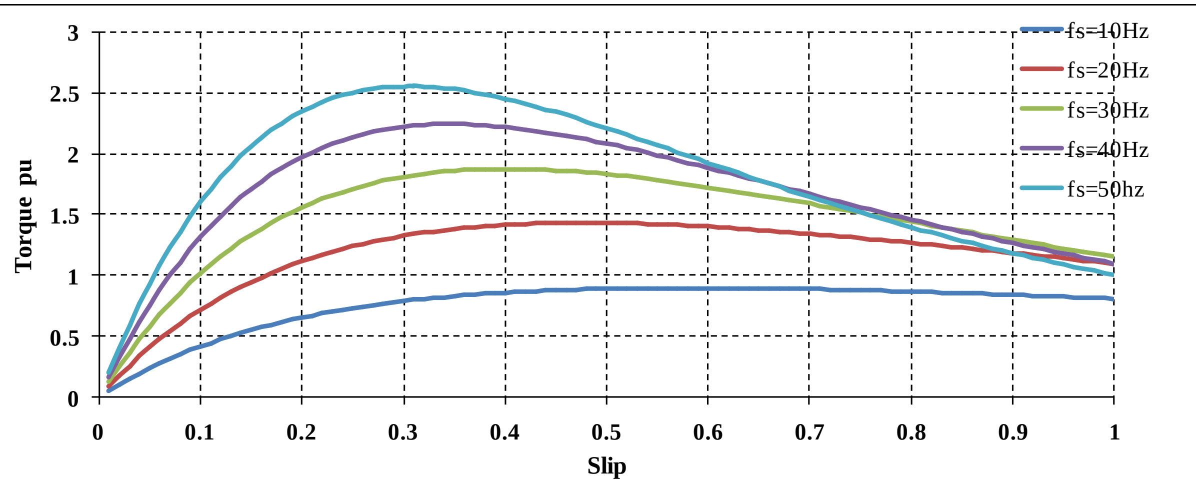

Figure 9: Torque vs. Slip Curve for Constant volts/Hz control of IM

References:

[1] R. Krishnan, *Electric motor drives: modeling, analysis, and control*, Prentice Hall, 2001

Lecture 20: Permanent magnet motor drives, their control and applications in EV/HEVs

Modeling of Induction Motor

Introduction

The topics covered in this chapter are as follows:

- Voltage Relations of Induction Motor
- Torque Equation in Machine Variables
- Equation of Transformation for Stator Variables
- Equation of Transformation for Rotor Variables
- Voltage and Torque Equations in Arbitrary Reference Frame Variables

Voltage Relations of Induction Motor

A 2 pole, 3 phase, Y connected symmetrical IM is shown in **Figure 1**. The stator windings are identical with N_s number of turns and the resistance of each phase winding is r_s. The rotor windings, may be wound or forged as squirrel cage winding, can be approximated as identical windings with equivalent turns N_r and resistance r_r. The air gap of IM is uniform and the stator and rotor windings are assumed to be sinusoidally distributed. The sinusoidal distribution of the windings results in sinusoidal magnetic field in the air gap.

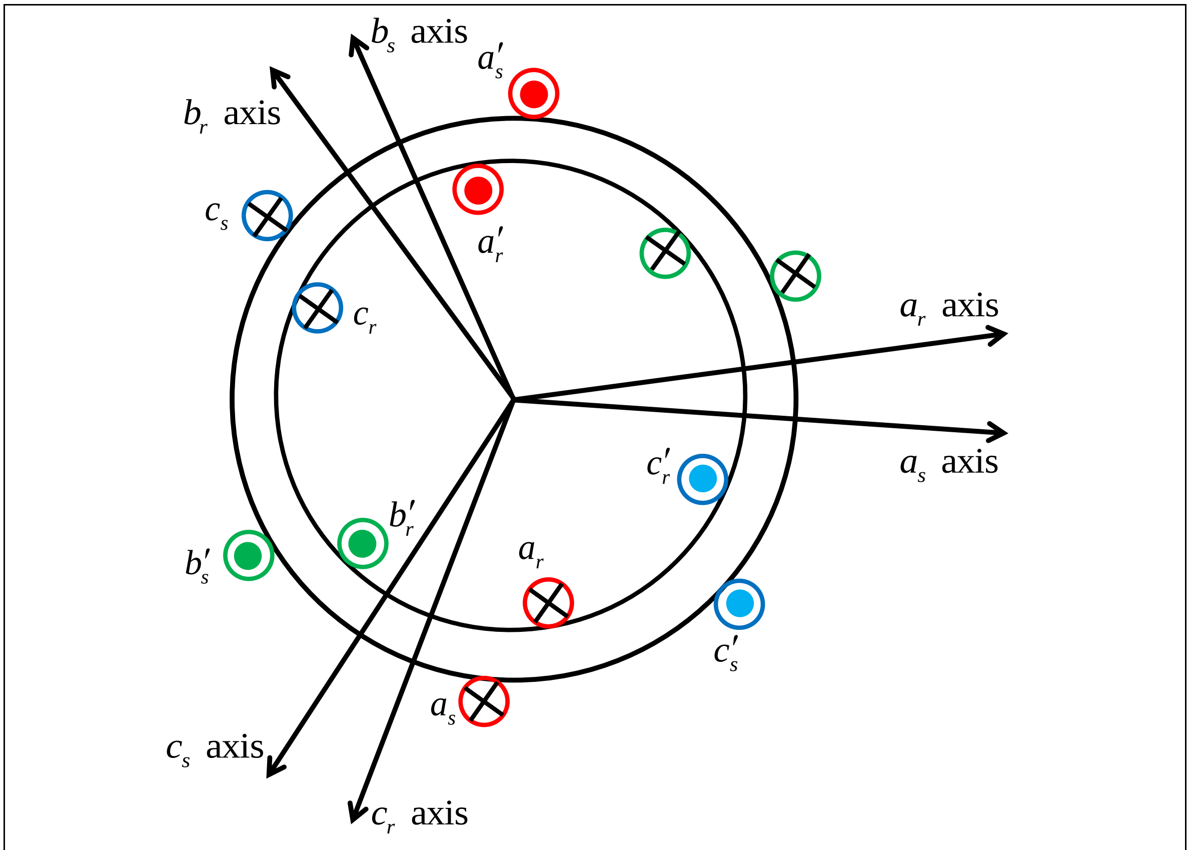

Figure 1: Two pole, 3 phase, wye-connected symmetrical induction motor

Since the windings are symmetric, the self inductances of stator windings are equal; that is

$$L_{asas} = L_{bsbs} = L_{cscs}$$
$$L_{asas} = L_{ls} + L_{ms} \tag{1}$$

where

L_{ls} is the leakage inductance of phase A or B or C of stator winding

L_{ms} is the stator magnetizing inductance

$L_{asas}, L_{bsbs}, L_{cscs}$ are the self inductances of stator phases

Similarly, the mutual inductances between the phases of the stator wind are same; that is

$$L_{asbs} = L_{bscs} = L_{csas} = -\frac{1}{2} L_{ms} \tag{2}$$

Based on the above discussion, the rotor self inductances and the mutual inductances between the rotor phases are

$$L_{arar} = L_{brbr} = L_{\mathrm{c}r\mathrm{c}r}$$
$$L_{arar} = L_{lr} + L_{mr} \tag{3}$$
where
L_{lr} is the leakage inductance of phase a of rotor winding
L_{mr} is the rotor magnetizing inductance

$$L_{arbr} = L_{brcr} = L_{\mathrm{c}rar} = -\frac{1}{2}L_{mr} \tag{4}$$

There exists mutual inductance between the stator and rotor windings. This mutual inductance is not constant because as the rotor rotates, the angle between the stator and rotor windings changes. Hence, the mutual inductance between the stator and rotor windings can be expressed as:

$$\begin{aligned} L_{asar} &= L_{bsbr} = L_{\mathrm{csc}r} = L_{sr}\cos(\theta_r) \\ L_{asbr} &= L_{bscr} = L_{csar} = L_{sr}\cos\left(\theta_r + \frac{2\pi}{3}\right) \\ L_{ascr} &= L_{bsar} = L_{csbr} = L_{sr}\cos\left(\theta_r - \frac{2\pi}{3}\right) \end{aligned} \tag{5}$$

The voltage equations for the IM shown in **Figure 1** are

$$\begin{aligned} v_{as} &= r_s i_{as} + \frac{d\lambda_{as}}{dt} \\ v_{bs} &= r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \\ v_{cs} &= r_s i_{cs} + \frac{d\lambda_{cs}}{dt} \\ v_{ar} &= r_r i_{ar} + \frac{d\lambda_{ar}}{dt} \\ v_{br} &= r_r i_{ar} + \frac{d\lambda_{ar}}{dt} \\ v_{cr} &= r_r i_{cr} + \frac{d\lambda_{cr}}{dt} \end{aligned} \tag{6}$$

The **equation 4** can be expressed in matrix form as

$$
\begin{aligned}
&\mathrm{v}_{abs} = \mathrm{r}_s \mathrm{i}_{abcs} + p\lambda_{abcs} \\
&\mathrm{v}_{abr} = \mathrm{r}_r \mathrm{i}_{abcr} + p\lambda_{abcr} \\
&\text{where} \\
&(\mathrm{v}_{abcs})^T = [v_{as} \quad v_{bs} \quad v_{cs}] \\
&(\mathrm{v}_{abcr})^T = [v_{ar} \quad v_{br} \quad v_{cr}] \\
&(\mathrm{i}_{abcs})^T = [i_{as} \quad i_{bs} \quad i_{cs}] \\
&(\mathrm{i}_{abcr})^T = [i_{ar} \quad i_{br} \quad i_{cr}] \\
&p = \frac{d}{dt}
\end{aligned}
\tag{7}
$$

In the above equation the subscript ***s*** refers to ***stator*** and ***r*** refers to ***rotor***. For a magnetically linear system, the flux linkages may be expressed as

$$
\begin{bmatrix} \lambda_{abcs} \\ \lambda_{abcr} \end{bmatrix} = \begin{bmatrix} \mathrm{L}_s & \mathrm{L}_{sr} \\ (\mathrm{L}_{sr})^T & \mathrm{L}_r \end{bmatrix} \begin{bmatrix} \mathrm{i}_{abcs} \\ \mathrm{i}_{abcr} \end{bmatrix} \tag{8}
$$

The stator and rotor windings inductances consist of self and mutual inductances and is represented as

$$
\mathrm{L}_s = \begin{bmatrix} L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} \end{bmatrix} \tag{9}
$$

$$
\mathrm{L}_r = \begin{bmatrix} L_{lr} + L_{mr} & -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & L_{lr} + L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} & L_{lr} + L_{mr} \end{bmatrix} \tag{10}
$$

$$
\mathrm{L}_{sr} = L_{sr} \begin{bmatrix} \cos(\theta_r) & \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) \\ \cos(\theta_r - \frac{2\pi}{3}) & \cos(\theta_r) & \cos(\theta_r + \frac{2\pi}{3}) \\ \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) & \cos(\theta_r) \end{bmatrix} \tag{11}
$$

The expression of voltage equation becomes convenient if all the rotor variables are referred to stator windings using the turns ratio:

$$\begin{aligned} \mathrm{i}'_{\mathrm{abcr}} &= \frac{N_r}{N_s}\mathrm{i}_{\mathrm{abcr}} \\ \mathrm{v}'_{\mathrm{abcr}} &= \frac{N_r}{N_s}v_{\mathrm{abcr}} \\ \lambda'_{\mathrm{abcr}} &= \frac{N_r}{N_s}\lambda_{\mathrm{abcr}} \end{aligned} \tag{12}$$

The magnetizing and mutual inductances are associated with the same magnetic flux path; hence,

$$L'_r = \left(\frac{N_s}{N_r}\right)^2 L_r;\ L'_{lr} = \left(\frac{N_s}{N_r}\right)^2 L_{lr}$$

$$\therefore \mathrm{L}'_r = \begin{bmatrix} L'_{lr}+L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L'_{lr}+L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L'_{lr}+L_{ms} \end{bmatrix} \tag{13}$$

Using **equation 13**, the flux linkages given by **equation 8** can be expressed as

$$\begin{bmatrix} \lambda_{\mathrm{abcs}} \\ \lambda'_{\mathrm{abcr}} \end{bmatrix} = \begin{bmatrix} \mathrm{L_s} & \mathrm{L}'_{\mathrm{sr}} \\ \left(\mathrm{L}'_{\mathrm{sr}}\right)^{\mathrm{T}} & \mathrm{L_r} \end{bmatrix} \begin{bmatrix} \mathrm{i}_{\mathrm{abcs}} \\ \mathrm{i}_{\mathrm{abcr}} \end{bmatrix} \tag{14}$$

and using **equation 14**, the voltage equation (**equation 7) can be written as**

$$\begin{bmatrix} \mathrm{v}_{\mathrm{abcs}} \\ \mathrm{v}_{\mathrm{abcr}} \end{bmatrix} = \begin{bmatrix} \mathrm{r_s}+\mathrm{pL_s} & \mathrm{pL}'_{\mathrm{sr}} \\ \mathrm{p}\left(\mathrm{L}'_{\mathrm{sr}}\right)^{\mathrm{T}} & \mathrm{r}'_{\mathrm{r}}+\mathrm{pL}'_{\mathrm{r}} \end{bmatrix} \begin{bmatrix} \mathrm{i}_{\mathrm{abcs}} \\ \mathrm{i}_{\mathrm{abcr}} \end{bmatrix}$$

$$\text{where } r'_r = \left(\frac{N_s}{N_r}\right)^2 r_r \tag{15}$$

Torque Equation in Machine Variables

The conversion into machine variables can be done using the principle of magnetic energy. In a machine, the stored magnetic energy is the sum of the self-inductance of each winding. The energy stored due to stator winding is:

$$W_s = \frac{1}{2}\left(\mathrm{i}_{\mathrm{abcs}}\right)^{\mathrm{T}}\left(\mathrm{L_s}\text{-}\mathrm{L_{ls}I}\right)\mathrm{i}_{\mathrm{abc}} \tag{16}$$

where

I is the identity matrix

Similarly, the energy stored due to rotor winding is

$$W_r = \frac{1}{2}\left(\mathrm{i}'_{\mathrm{abcr}}\right)^{\mathrm{T}}\left(\mathrm{L}'_r\text{-}\mathrm{L}'_{\mathrm{lr}}\mathrm{I}\right)\mathrm{i}'_{\mathrm{abcr}} \tag{17}$$

The energy stored due to mutual inductance between the stator and rotor windings is

$$W_{sr} = (\mathrm{i}_{\mathrm{abcs}})^{\mathrm{T}}\, \mathrm{L}'_{sr}\mathrm{i}'_{\mathrm{abcr}} \tag{18}$$

Hence, the total energy stored in the magnetic circuit of the motor is

$$\begin{aligned} W_f &= W_s + W_{sr} + W_r \\ &= \frac{1}{2}(\mathrm{i}_{\mathrm{abcs}})^T (\mathrm{L_s}\text{-}\mathrm{L_{ls}}\mathrm{I})\mathrm{i}_{\mathrm{abc}} + (\mathrm{i}_{\mathrm{abcs}})^T \mathrm{L}'_{sr}\mathrm{i}'_{\mathrm{abcr}} + \frac{1}{2}(\mathrm{i}'_{\mathrm{abcr}})^T (\mathrm{L}'_r\text{-}\mathrm{L}'_{\mathrm{lr}}\mathrm{I})\mathrm{i}'_{\mathrm{abcr}} \end{aligned} \tag{19}$$

Since the magnetic circuit of the machine is assumed to be linear (the of **B** vs. **H**), the stored energy in the magnetic field W_f is equal to the co-energy W_{co}. The electromagnetic torque produced by the IM is given by

$$T_e = \left(\frac{P}{2}\right)\frac{\partial W_{co}}{\partial \theta_r} = \left(\frac{P}{2}\right)\frac{\partial W_f}{\partial \theta_r} \tag{20}$$

where θ_r is the rotor angle at any given point of time.

Since the inductances L_s, L_{ls}, L'_r, L'_{lr} are not functions of θ_r and only L'_{sr} is a function of θ_r (**equation 11**), substituting the **equation 19** into **equation 21** gives

$$T_e = \left(\frac{P}{2}\right)(\mathrm{i}_{\mathrm{abcs}})^T \frac{\partial}{\partial \theta_r}[\mathrm{L}'_{\mathrm{sr}}]\mathrm{i}'_{\mathrm{abcr}} \tag{21}$$

Linear Transformations

From **equation 6** it can be seen that in order to study the dynamic behaviour of IM, a set of ***six*** equations have to be solved. If the number of equations to be solved is reduced, the computational burden will be reduced. In order to reduce the number of equations ***linear transformation*** is carried out. It is very common to use linear transformation to solve problems and one of the most common examples of it is ***logarithm***. The logarithms are used to multiply or divide two numbers. Similarly, the Laplace transform is also a linear transformation. It transforms the time-domain equations to $s-$domain equation and after manipulations, one again gets the required time-domain solution. The process of referring secondary quantities to primary or primary to secondary in a transformer is also equivalent to a linear transformation. It should be noted that ***the transformation from old to new set of variables is carried out for simplifying the calculations***.

Linear transformations in electrical machines are usually carried out to obtain new equations which are fewer in number or are more easily solved. For example, a three phase machine are more complicated because of the magnetic coupling amongst the three phase windings as seen from **equation 11**, but this is not the case after the transformation.

Transformation from Three Phases to Two Phases (a, b, c to $\alpha, \beta, 0$)

A symmetrical 2 pole, 3 phase winding on the rotor is represented by three coils ***A, B, C*** each of N_r turns and displaced by 120^o is shown in **Figure 2**.

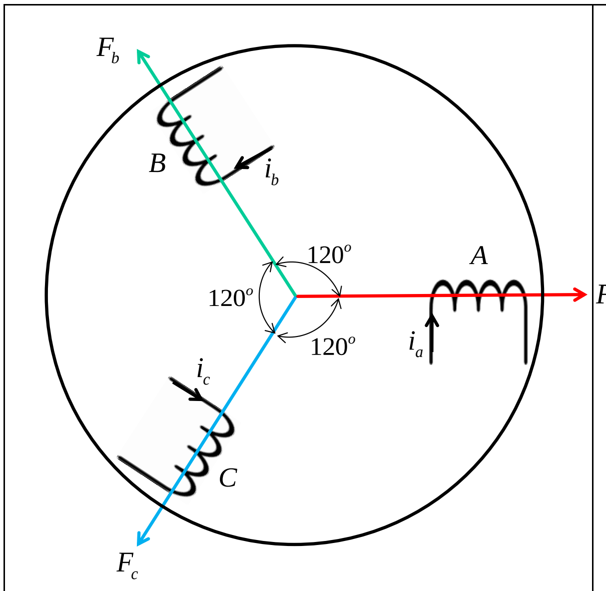

Figure 2: A symmetrical 2 pole 3phase winding on the rotor

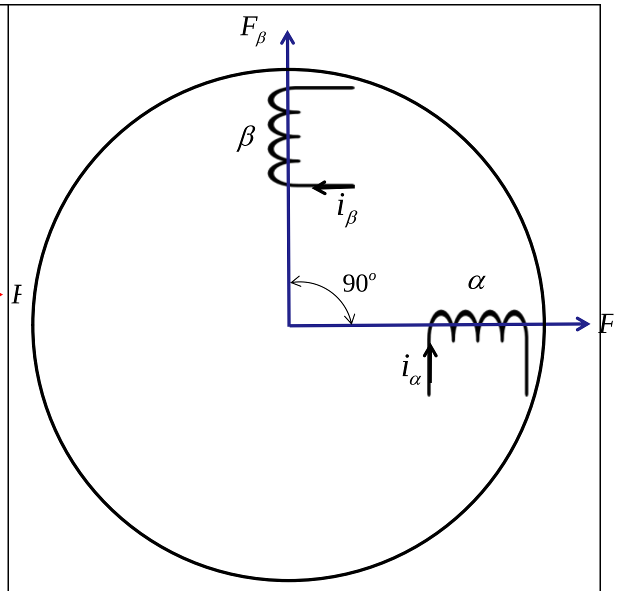

Figure 3: A balanced 2phase winding of the rotor

Maximum values of mmfs F_a, F_b, F_c are shown along their respective phase axes. The combined effect of these three mmfs results in mmf of constant magnitude rotating at a constant angular velocity depending on the poles and frequency. If the three currents in the rotor are:

$$i_a = I_m \cos(\omega t);\; i_b = I_m \cos\left(\omega t - \frac{2\pi}{3}\right);\; i_c = I_m \cos\left(\omega t - \frac{4\pi}{3}\right) \tag{22}$$

The currents given in **equation 22** produce a mmf of constant magnitude $\frac{3I_m N_r}{2}$ rotating with respect to three phase winding at the frequency of ω. In **Figure 3**, a balanced two phase winding is represented by two orthogonal coils α, β on the rotor. For the sake of convenience the axes of phase ***A*** and α are taken to be coincident. The two phase currents flowing in the winding is given by

$$i_\alpha = I_m \cos(\omega t);\; i_\beta = I_m \cos\left(\omega t - \frac{\pi}{2}\right) \tag{23}$$

These two phase currents result in a mmf of constant magnitude $I_m N_r$ rotating with respect to the two phase windings at the frequency of the currents. The mmf of three phase and two phase systems can be rendered equal in magnitude by making any one of the following changes:

i. By changing the magnitude of the two phase currents
ii. By changing the number of turns of the two phase windings
iii. By changing both the magnitude of currents and number of turns

In the following subsections each of the three cases are discussed.

Changing the magnitude of two phase currents

In this case the number of turns in the two phase winding is N_r which is same as that of the three phase windings. Hence, in order to have equal mmf, the new magnitude of the current in the two phases must be determined. To obtain the new values of the two phase currents the instantaneous three phase mmfs are resolved along the α axis shown in **Figure 3**:

$$i_\alpha N_r = \left(i_a \cos 0 + i_b \cos\frac{2\pi}{3} + i_c \cos\frac{4\pi}{3} \right) N_r \Rightarrow i_\alpha = i_a - \frac{1}{2}\left(i_b + i_c\right) \tag{24}$$

Similarly, the resolving the three phase currents along the β axis gives

$$i_\beta N_r = \left(i_a \sin 0 + i_b \sin\frac{2\pi}{3} + i_c \sin\frac{4\pi}{3} \right) N_r \Rightarrow i_\beta = \left(\frac{\sqrt{3}}{2} i_b - \frac{\sqrt{3}}{2} i_c \right) \tag{25}$$

For a balanced three phase system the sum of three currents is zero, that is

$$i_a + i_b + i_c = 0 \tag{26}$$

Using **equation 26** into **equation 24** gives

$$i_\alpha = \frac{3}{2} i_a \tag{27}$$

Substituting the values of i_a, i_b and i_c from **equation 22** into **equations 25** and **27** gives

$$i_\alpha = \frac{3}{2} I_m \cos\left(\omega t\right);\; i_\beta = \frac{3}{2} I_m \sin\left(\omega t\right) \tag{28}$$

From **equation28** it can be seen that the magnitude of the two phase currents is 3/2 times the magnitude of the three phase currents. Since the number of turns per phase is same in both the three and two phase windings, the magnitude of phase e.m.fs of the two and three phase windings would be equal. The power per phase of the two phase system is $3/2VI_m$ and the power per phase of a three phase winding is VI_m. However, the total power produced by a two phase system is $(=2\cdot3/2\cdot VI_m=3VI_m)$ and that produced by a three phase system is $3VI_m$. Thus, the linear transformation is power invariant. The only disadvantage is that the transformation of current and voltage will differ because of presence of factor 3/2 in the current transformation. As factor 3/2 appears in current transformation and not in voltage transformation, the per phase parameters of the two phase and three phase machine will not be the same.

Changing the number of turns of two phase winding

If the number of turns of two phase winding is made 3/2 times that of the three phase winding, then for equal mmfs the following relation between the two phase and three phase currents holds:

$$\frac{3}{2}i_\alpha N_r=\left(i_a\cos 0+i_b\cos\frac{2\pi}{3}+i_c\cos\frac{4\pi}{3}\right)N_r\Rightarrow\frac{3}{2}i_\alpha=i_a-\frac{1}{2}\left(i_b+i_c\right)=\frac{3}{2}i_a\Rightarrow i_\alpha=i_a \tag{29}$$

$$\frac{3}{2}i_\beta N_r=\left(i_a\sin 0+i_b\sin\frac{2\pi}{3}+i_c\sin\frac{4\pi}{3}\right)N_r\Rightarrow i_\beta=\frac{2}{3}\left(\frac{\sqrt{3}}{2}i_b-\frac{\sqrt{3}}{2}i_c\right) \tag{30}$$

Substituting the values of i_a, i_b and i_c from **equation 22** into **equations 29** and **30** gives

$$i_\alpha=I_m\cos\left(\omega t\right);\ i_\beta=I_m\sin\left(\omega t\right) \tag{31}$$

Since, the number of turns in the two phase winding is 3/2 times that of three phase winding, the per phase voltage of the two phase machine will be 3/2 times the per phase voltage of the three phase systems. Hence,

The power per phase in two phase system= $\frac{3}{2}VI_m$

Total in two phase system= $3VI_m$

The power per phase in three phase system= VI_m

Total in three phase system= $3VI_m$

Here again the power invariance is obtained, but, as in the previous case, the transformation of current and voltage will differ because of the factor 3/2 in the voltage transformation. In this case the per phase parameters of the machine will be different for two and three phase systems.

Changing both the number of turns and magnitude of current of two phase winding

In this case both the magnitude of currents and number of turns of the two phase system are changed to obtain identical transformation for voltage and current. To do so the number of turns in the two phase winding is made $\sqrt{3/2}$ times that of three phase winding. Then for equal m.m.f the following holds

$$\sqrt{\frac{3}{2}} i_\alpha N_r = \left(i_a \cos 0 + i_b \cos\frac{2\pi}{3} + i_c \cos\frac{4\pi}{3} \right) N_r$$

$$\Rightarrow i_\alpha = \sqrt{\frac{2}{3}} \left(i_a - \frac{1}{2} i_b - \frac{1}{2} i_c \right) \Rightarrow i_\alpha = \sqrt{\frac{3}{2}} I_m \cos(\omega t)$$

(32)

$$\sqrt{\frac{3}{2}} i_\beta N_r = \left(i_a \sin 0 + i_b \sin\frac{2\pi}{3} + i_c \sin\frac{4\pi}{3} \right) N_r$$

$$\Rightarrow i_\beta = \sqrt{\frac{2}{3}} \left(0 + \frac{\sqrt{3}}{2} i_b - \frac{\sqrt{3}}{2} \frac{1}{2} i_c \right) \Rightarrow i_\beta = \sqrt{\frac{3}{2}} I_m \sin(\omega t) \qquad (33)$$

Since the number of turns in the two phase winding is $\sqrt{\frac{3}{2}}$ times that of three phase winding, the voltage per phase of the two phase winding is $\sqrt{\frac{3}{2}}$ times that of the three phase winding. Hence, the phase voltage and current of the two phase system are $\sqrt{\frac{3}{2}}$ times that of three phase system. This results in identical transformations for both the voltage and current and the per phase quantities of the machine, such as the impedance per phase, will be same for two and three phase systems.

Hence, the transformation equations for converting three phase currents into two phase currents, given by **equations 32** and **33**, can be expressed in matrix form as

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \qquad (34)$$

The transformation matrix in **equation 34** is singular and hence i_a, i_b and i_c cannot be obtained from i_α, i_β. The matrix can be made square matrix if a third equation of constraint between i_a, i_b and i_c is introduced. Since, the magnitude and direction of the mmf produced by two and three phase systems are identical, the third current in terms of i_a, i_b and i_c should not produce any resultant air gap mmf. Hence, a zero sequence current is introduced and it is given by

$$i_0 = \frac{1}{\sqrt{3}}\left(i_a + i_b + i_c\right) \tag{35}$$

Due to the fact that sum of three phase currents in a balanced system is zero (**equation 26**), the zero sequence current does not produce any rotating mmf. Using the **equation 35** the matrix representation given in **equation 34** can be written as

$$\begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \tag{36}$$

The transformation matrix now is non-singular and its inverse can be easily obtained.

Transformation from Rotating Axes ($\alpha,\beta,0$) to Stationary Axes ($d,q,0$)

In **Figure 3**, three phase and two phase windings are shown on the rotor and hence both the windings rotate at the same speed and in the same direction. Hence, both the two phase and three phase windings are at rest with respect to each other.

In this section the transformation of rotating $\alpha,\beta,0$ quantities to stationary $d,q,0$ quantities is carried out. When transformation is carried out from rotating to stationary axes, the relative position of rotating axes varies with respect to stationary or fixed axes. Hence, the transformation matrix must have coefficients that are functions of the relative position of the moving α,β and fixed d,q axes. In **Figure 4** the rotating α,β axes are shown inside the circle and the stationary d,q axes are shown outside. The angle θ_r shown in **Figure 4** is such that at time $t=0$, $\theta_r=0$, that is, the α,β axis is aligned with the d,q axis.

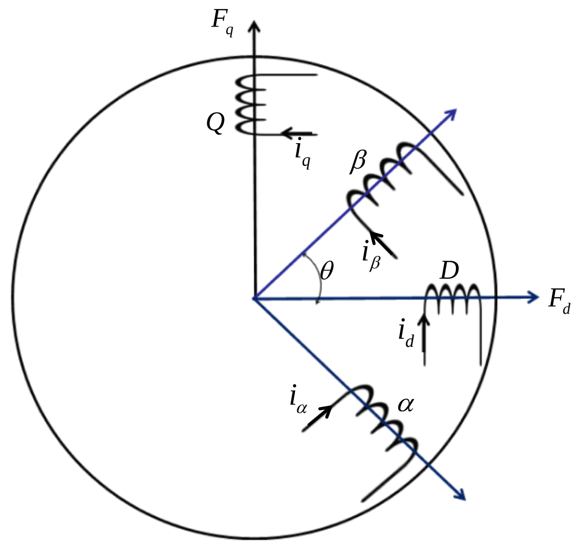

Figure 4: The *d* and *q* axis on the rotor

At any time t, $\theta_r=\omega_r t$, where ω_r is the angular speed of the rotor. Assuming same number of turns in the α,β and d,q windings, the mmfs F_α and F_β can be resolved along the d,q axis as

$$\begin{aligned}
&F_d = F_\alpha \cos\theta_r + F_\beta \sin\theta_r \Rightarrow N_r i_d = N_r i_\alpha \cos\theta_r + N_r i_\beta \sin\theta_r \\
&\Rightarrow i_d = i_\alpha \cos\theta_r + i_\beta \sin\theta_r \\
&i_q = -i_\alpha \sin\theta_r + i_\beta \cos\theta_r
\end{aligned} \tag{37}$$

The **equation** 37 can be expressed in the matrix form as

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos\theta_r & \sin\theta_r \\ -\sin\theta_r & \cos\theta_r \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}$$

and

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} \tag{38}$$

Let the currents in d,q axis winding be

$$\begin{aligned} i_d &= I_m \sin(\omega t + \phi) \\ i_q &= I_m \cos(\omega t + \phi) \end{aligned} \tag{39}$$

where

ϕ is a constant arbitrary phase angle

Using **equation 38** and **equation 39**, the currents i_α, i_β are obtained as

$$\begin{aligned} i_\alpha &= I_m \sin(\theta - \theta_r + \phi) \\ i_\beta &= I_m \cos(\theta - \theta_r + \phi) \\ \Rightarrow i_\alpha &= I_m \sin(\omega t - \omega_r t + \phi) \\ i_\beta &= I_m \cos(\omega t - \omega_r t + \phi) \end{aligned} \tag{40}$$

where

$\theta = \omega t$

In case the frequency of the d and q axis current is same as the speed of rotation of the rotor, then

$$\begin{aligned} i_\alpha &= I_m \sin(\phi) \\ i_\beta &= I_m \cos(\phi) \end{aligned} \tag{41}$$

Thus, time varying currents in stationary d,q axis result in mmf which is identical to the mmf produced by constant currents (or d.c.) in rotating α,β axis.

In the above transformation the zero sequence current is not transformed and it can be taken into account by an additional column in the **equation 38**.

$$\begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} = \begin{bmatrix} \cos\theta_r & \sin\theta_r & 0 \\ -\sin\theta_r & \cos\theta_r & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_o \end{bmatrix} \tag{42}$$

Substituting the values of i_α, i_β, i_0 from **equation 32** into **equation 42** gives

$$\begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta_r & \cos\left(\theta_r - \frac{2\pi}{3}\right) & \cos\left(\theta_r + \frac{2\pi}{3}\right) \\ -\sin\theta_r & -\sin\left(\theta_r - \frac{2\pi}{3}\right) & -\sin\left(\theta_r + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$\left(i_d \quad i_q \quad i_0\right)^T = K_s \left(i_a \quad i_b \quad i_c\right)^T$$

where (43)

$$K_s = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta_r & \cos\left(\theta_r - \frac{2\pi}{3}\right) & \cos\left(\theta_r + \frac{2\pi}{3}\right) \\ \sin\theta_r & \sin\left(\theta_r - \frac{2\pi}{3}\right) & \sin\left(\theta_r + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The inverse transformation matrix is given by

$$K_S^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta_r & -\sin\theta_r & \frac{1}{\sqrt{2}} \\ \cos\left(\theta_r - \frac{2\pi}{3}\right) & -\sin\left(\theta_r - \frac{2\pi}{3}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\theta_r + \frac{2\pi}{3}\right) & -\sin\left(\theta_r + \frac{2\pi}{3}\right) & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{44}$$

The above transformation is valid for any electrical quantity such as current, voltage, flux linkage, etc. In general the three phase voltages, currents and fluxes can be converted into dqo phases using the following transformation matrices

$$\left(\mathrm{f_{qdo}}\right)^{\mathrm{T}} = \mathrm{K_s}\left(\mathrm{f_{abc}}\right)^{\mathrm{T}}$$

where (45)

$$\left(\mathrm{f_{qdo}}\right)^{\mathrm{T}} = \left[f_q \quad f_d \quad f_o\right]; \ \left(\mathrm{f}_{abc}\right)^{\mathrm{T}} = \left[f_a \quad f_b \quad f_c\right]$$

Transformation of Induction Motor Quantities into Stationary d,q Axis

The three phase stator and rotor wind axis are shown in **Figure 5a**. In **Figure 5a**, the subscript ***s*** represents stator quantities and ***r*** represents rotor quantities. For the stator α,β axis and d,q axis are coincident as shown in **Figure 5b** and hence there is no difference between α,β and d,q stator quantities. Examination of **Figure 5a** and **5b** reveals that phase A_s coincides with the phase α axis or phase d axis of the 2 phase machines. As a result of this, the results obtained for d axis quantities apply equally well to the α phase of the 2 phase machine. The conversion of 3 phase stator winding to 2 phase stator winding is given by **equation 36** as

$$\begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_{0s} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix} \tag{46}$$

Since the α,β and d,q axis both lie on the stator are stationary with respect to each other, the transformation from α,β to d,q axis is given by

$$\begin{bmatrix} i_{ds} \\ i_{qs} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix} \tag{47}$$

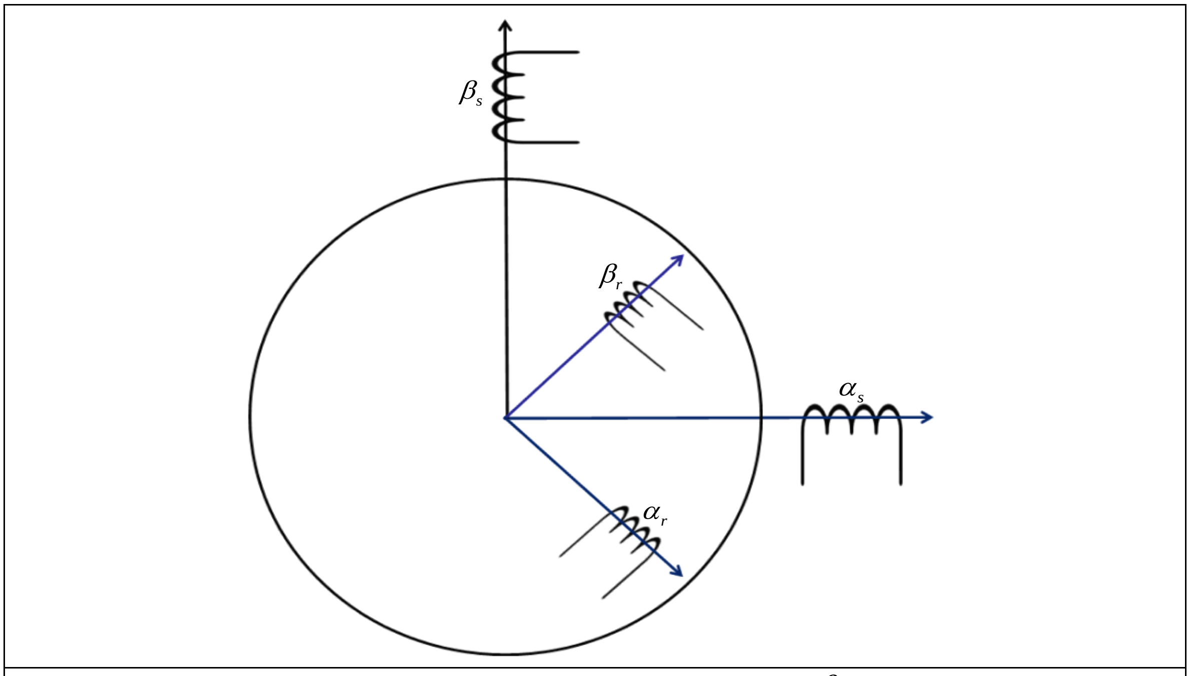

Figure 5a: Equivalent 2 phase induction machine in α,β axis

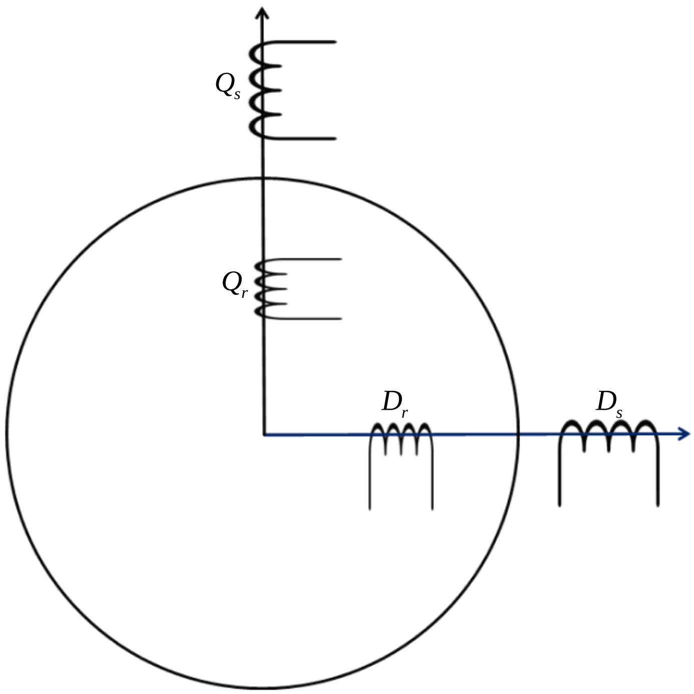

Figure 5b: Equivalent 2 phase induction machine in d,q axis

In case of rotor currents, the transformation from 3 phase to 2 phase is given by

$$\begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \\ i_{0r} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix} \tag{48}$$

The transformation 2 phase rotor currents from rotating α,β to stationary d,q axis is given by

$$\begin{bmatrix} i_{dr} \\ i_{qr} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} i_{\alpha r} \\ i_{\beta r} \end{bmatrix} \tag{49}$$

The rotor voltages are related by identical expressions. The frequency of the currents in 3phase rotor windings or in 2phase rotor windings is the slip frequency sf. Since the d,q axes windings are stationary, the frequency of rotor currents i_{dr} and i_{qr} is the line frequency f.

The Machine Performance Equations

The general voltage equation for the machine shown in **Figure 5b** is

$$\begin{bmatrix} v_{ds} \\ v_{qs} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_{ds}+L_{ds}p & 0 & M_d p & 0 \\ 0 & r_{qs}+L_{qs}p & 0 & M_q p \\ M_d p & -M_q\omega_r & r_{dr}+L_{dr}p & -L_{qr}\omega_r \\ M_d\omega_r & M_d p & L_{dr}\omega_r & r_{qr}+L_{qr}p \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i_{dr} \\ i_{qr} \end{bmatrix} \quad (50)$$

where $p = \dfrac{d}{dt}$

The following points about the IM are to be considered:

- Stator and rotor both have balanced winding configurations, hence:
  $r_{ds} = r_{qs} = r_s$ =resistance of each stator coil
  $r_{dr} = r_{qr} = r_r$ =resistance of each rotor coil
- Since the air gap is uniform, the self inductances of d and q axis of the stator winding are equal and that of the rotor windings are also equal, that is
  $L_{ds} = L_{qs} = L_s$ =self inductance of the stator winding
  $L_{dr} = L_{qr} = L_r$ =self inductance of the rotor winding
- The d and q axis coils are identical, the mutual inductance between the stator and rotor d axis coils is equal to the mutual inductance between stator and rotor q axis coils, that is
  $M_d = M_q = L_m$

In an induction machine the rotor windings are short circuited, therefore no emf exists in the winding of the rotor and $v_{dr} = v_{qr} = 0$. Since the rotor winding is short circuited, the direction of i_{dr} and i_{qr} are reversed and this has to be taken into account in the general voltage equations.

Based on the above discussion, the general voltage equation becomes:

$$\begin{bmatrix} v_{ds} \\ v_{qs} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_s+L_s p & 0 & -L_m p & 0 \\ 0 & r_s+L_s p & 0 & -L_m p \\ L_m p & -L_m\omega_r & -(r_r+L_r p) & L_r\omega_r \\ L_m\omega_r & L_m p & -L_r\omega_r & -(r_r+L_r p) \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i_{dr} \\ i_{qr} \end{bmatrix} \quad (51)$$

The voltage equations given in **equation 51** can be written as:

$V=[R]I+[L]pI+[G]\omega_r I$

where

$$V=\begin{bmatrix} v_{ds} \\ v_{qs} \\ v_{dr} \\ v_q \end{bmatrix};\ I=\begin{bmatrix} i_{ds} \\ i_{qs} \\ i_{dr} \\ i_q \end{bmatrix};\ \left[R\right]=\begin{bmatrix} r_s & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 \\ 0 & 0 & r_s & 0 \\ 0 & 0 & 0 & r_s \end{bmatrix} \quad (52)$$

$$\left[L\right]=\begin{bmatrix} L_s & 0 & L_m & 0 \\ 0 & L_s & 0 & L_m \\ L_m & 0 & L_r & 0 \\ 0 & L_m & 0 & L_r \end{bmatrix};\left[G\right]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -L_m & 0 & L_r \\ L_m & 0 & -L_r & 0 \end{bmatrix}$$

The instantaneous input power to the machine is given by

$$P=I^T V=I^T\left[R\right]I+I^T\left[L\right]pI+I^T\left[G\right]\omega_r I \quad (53)$$

In **equation 53** the term $I^T\left[R\right]I$ represents the stator and rotor resistive losses. The term $I^T\left[L\right]pI$ denotes the rate of change of stored magnetic energy. Hence, what is left of the power component must be equal to the air gap power given by the term $I^T\left[G\right]\omega_r I$. The air gap torque is given by

$$\omega_m T_e=I^T\left[G\right]\omega_r I$$

$$\omega_r=\frac{N_p}{2}\omega_m$$

$$\Rightarrow T_e=\frac{N_p}{2}I^T\left[G\right]I \quad (54)$$

where

ω_m is the mechanical speed of the rotor and

N_p is the number of poles

Substituting the value of $\left[G\right]$ from **equation52** into **equation 54** gives

$$T_e=\frac{N_p}{2}L_m\left(i_{qs}i_{dr}-i_{ds}i_{qr}\right) \quad (55)$$

The torque given by **equation 55** can also be written as

$$T_e=\frac{N_p}{2}L_m\left(\lambda_{qr}i_{dr}-\lambda_{dr}i_{qr}\right) \quad (56)$$

References:

[1] R. Krishnan, *Electric motor drives: modeling, analysis, and control*, Prentice Hall, 2001

[2] P. C. Krause, O. Wasynczuk, S. D. Sudhoff, *Analysis of electric machinery*, IEEE Press, 1995

Lecture 21: Switch reluctance motors, their configurations and optimization

Field Oriented Control of Induction Motor

Introduction

The topics covered in this chapter are as follows:

- Field Oriented Control (FOC)
- Direct Rotor Oriented FOC
- Indirect Rotor Oriented FOC

Field Oriented Control (FOC)

In an Electric Vehicle, it is required that the traction motor is able to deliver the required torque almost instantaneously. In an induction motor (IM) drive, such performance can be achieved using a class of algorithms known as ***Field Oriented Control*** *(FOC)*. There are varieties of FOC such as:

- Stator flux oriented
- Rotor flux oriented
- Air gap flux oriented

Each of the above mentioned control method can be implemented using ***direct*** or ***indirect*** methods.

The basic premise of FOC may be understood by considering the current loop in a uniform magnetic field as shown in **Figure 1a**. From Lorenz force equation, it can be seen that the torque acting on the current loop is given by

$$T_e = -2BiNLr\sin\theta \tag{1}$$

where

B is the flux density

i is the current

N is the number of turns

L is the length of the coil

r is the radius of the coil

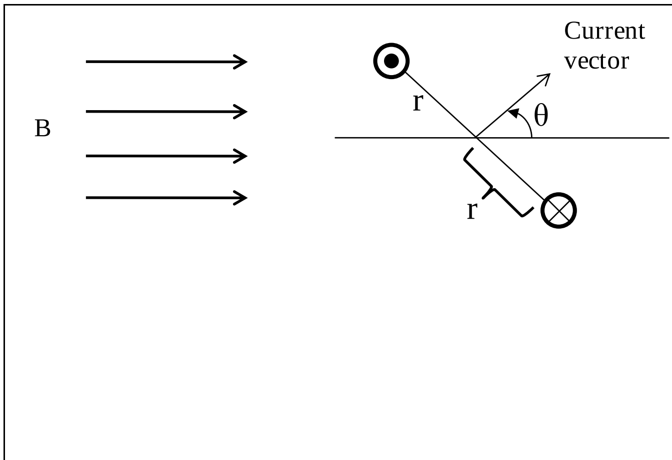

Figure 1a: A coil in a magnetic field

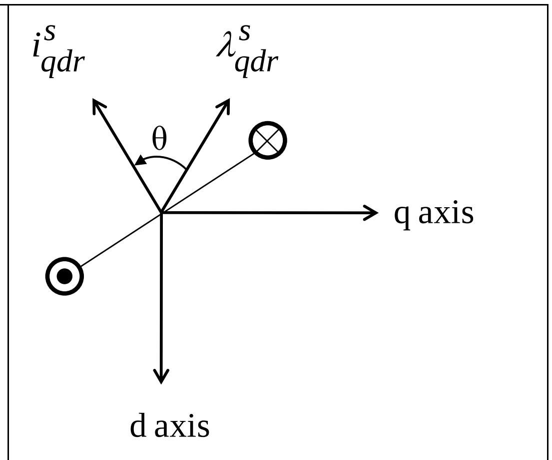

Figure 1b: Orientation of magnetic field in IM

From **equation 1** it is evident that the torque is maximised when the current vector is perpendicular to the magnetic field. The same conclusion can be applied to an IM. In **Figure 1b** orientations of magnetic fields and currents in an IM are shown. The rotor current and flux linkage vectors are shown in **Figure 1** at some instant of time. Hence, the torque produced by the motor (refer to Lecture 19) is given by

$$T_e = \frac{3}{2}\frac{P}{2}\left(\lambda'_{qr} i'_{dr} - \lambda'_{dr} i'_{qr}\right) \tag{2}$$

The **equation 2** can be re-written as

$$T_e = -\frac{3}{2}\frac{P}{2}\left|\lambda'_{qdr}\right|\left|i'_{qdr}\right|\sin\theta \tag{3}$$

The **equation 3** is analogous to **equation 1**. Hence, for a given magnitude of flux linkage, torque is maximised when the flux linkage and current vectors are perpendicular. Therefore, it is desirable to keep the rotor flux linkage perpendicular to rotor current vector.

In the analysis of FOC the following convention will be used:

- The parameters with a superscript "s" are in stator frame of reference.
- The parameters with a superscript "e" are in synchronous frame of reference.
- The parameters with subscript "r" indicate rotor parameters.
- The parameters with subscript "s" indicate stator parameters.
- All rotor quantities are referred to stator using the turns ratio of the windings (Lecture 17) and hence "$'$" is dropped.

In case of singly excited IMs (*in singly excited IM, the rotor winding is not fed by any external voltage source. In case of wound rotor machines, they are short circuited using slip rings. For cage IMs, the rotor bars are short circuited at the terminals*), the rotor flux linkage vector and rotor current vector are always perpendicular. The voltage equations for the IM (refer to Lecture 19) in synchronous frame of reference are

$$\begin{aligned}
v_{qs}^e &= r_s i_{qs}^e + \omega_e \lambda_{ds}^e + p\lambda_{qs}^e \\
v_{ds}^e &= r_s i_{ds}^e - \omega_e \lambda_{qs}^e + p\lambda_{ds}^e \\
v_{os}^e &= r_s i_{os}^e + p\lambda_{os}^e \\
v_{qr}^e &= r_r i_{qr}^e + \left(\omega_e - \omega_r\right)\lambda_{dr}^e + p\lambda_{qr}^e \\
v_{dr}^e &= r_r i_{dr}^e - \left(\omega_e - \omega_r\right)\lambda_{qr}^e + p\lambda_{dr}^e \\
v_{or}^e &= r_r i_{or}^e + p\lambda_{or}^e
\end{aligned} \tag{1}$$

where

ω_e is the rotational speed of synchronous frame of reference

In case of singly excited IM, the rotor voltages are zero, that is $v_{qr}^e = 0$, $v_{dr}^e = 0$ and $v_{or}^e = 0$. Hence, the rotor currents can be obtained as

$$\begin{aligned}
0 &= r_r i_{qr}^e + \left(\omega_e - \omega_r\right)\lambda_{dr}^e + p\lambda_{qr}^e \Rightarrow i_{qr}^e = -\frac{1}{r_r}\left(\omega_e - \omega_r\right)\lambda_{dr}^e - p\lambda_{qr}^e \\
0 &= r_r i_{dr}^e - \left(\omega_e - \omega_r\right)\lambda_{qr}^e + p\lambda_{dr}^e \Rightarrow i_{dr}'^e = \frac{1}{r_r}\left(\omega_e - \omega_r\right)\lambda_{qr}^e - p\lambda_{dr}^e \\
0 &= r_r i_{or}^e + p\lambda_{or}^e \Rightarrow i_{or}^e = -\frac{p\lambda_{or}^e}{r_r}
\end{aligned} \tag{2}$$

Since steady state operation of IM is considered, the time derivative term of flux linkage in **equation 2** will vanish. Hence, the rotor currents are:

$$\begin{aligned}
i_{qr}^e &= -\frac{1}{r_r}\left(\omega_e - \omega_r\right)\lambda_{dr}^e \\
i_{dr}^e &= \frac{1}{r_r}\left(\omega_e - \omega_r\right)\lambda_{qr}^e \\
i_{or}^e &= 0
\end{aligned} \tag{3}$$

The dot product of the rotor flux linkage and rotor current vectors may be expressed as

$$\lambda_{qdr}^e . i_{qdr}^e = \lambda_{qr}^e . i_{qr}^e + \lambda_{dr}^e . i_{dr}^e \tag{4}$$

Substituting the values of i_{qr}^{e} and i_{dr}^{e} from **equation 3** into **equation 4** gives

$$\lambda_{qdr}^{e}.i_{qdr}^{e}=-\frac{\lambda_{qr}^{e}}{r_r}\left(\omega_e-\omega_r\right)\lambda_{dr}^{e}+\frac{\lambda_{dr}^{e}}{r_r}\left(\omega_e-\omega_r\right)\lambda_{qr}^{e}=0 \tag{5}$$

Form **equation 5** it can be seen that the dot product between the rotor flux and rotor current vectors is zero in case of singly excited IM. Hence, it can be concluded that the rotor flux and rotor current vectors are perpendicular to each other in steady state operation. The defining feature of FOC is that this characteristic (that the rotor flux and rotor current vectors are perpendicular to each other) is maintained during transient conditions as well.

In both direct and indirect FOC, the 90^{o} shift between the rotor flux and rotor current vector can be achieved in two steps:

- The first step is to ensure that

  $$\lambda_{qr}^{e}=0 \tag{6}$$

- The second step is to ensure that

  $$i_{dr}^{e}=0 \tag{7}$$

By suitable choice of θ_s on an instantaneous basis, **equation 6** can be achieved. Satisfying **equation 7** can be accomplished by forcing d-axis stator current to remain constant. To see this, consider the d-axis rotor voltage equation

$$0=r_r i_{dr}^{e}+\left(\omega_e-\omega_r\right)\lambda_{qr}^{e}+p\lambda_{dr}^{e} \tag{8}$$

Since $\lambda_{qr}^{e}=0$, **equation 8** can be written as

$$0=r_r i_{dr}^{e}+p\lambda_{dr}^{e} \tag{9}$$

The d-axis rotor flux linkage is given by (refer Lecture 19):

$$\lambda_{dr}^{e}=L_{lr}i_{dr}^{e}+L_m\left(i_{ds}^{e}+i_{dR}^{e}\right) \tag{10}$$

Substituting the value of λ_{dr}^{e} from **equation 10** into **equation 9** gives:

$$pi_{dr}^{e}=-\frac{r_r}{L_{lr}}i_{dr}^{e}-\frac{L_m}{L_{lr}}pi_{ds}^{e} \tag{11}$$

If i_{ds}^{s} is held constant, then $pi_{ds}^{e}=0$ and the solution of **equation 11** becomes

$$i_{dr}^{e}=Ce^{-\left(\frac{r_r'}{L_{lr}}\right)t} \tag{12}$$

where

C is a constant of integration

It is evident from **equation 12** that the rotor current i_{dr}^{e} will decay to zero and stay at zero regardless of other transients that may be taking place. Hence, the torque (from **equation 2**) is given by

$$T_e = \frac{3}{2}\frac{P}{2}\lambda_{dr}^{e} i_{qr}^{e} \qquad (13)$$

The q-axis rotor flux is given by (refer Lecture 19)

$$\lambda_{qr}^{e} = L_{lr} i_{qr}^{e} + L_m \left(i_{qs}^{e} + i_{qr}^{e} \right) \qquad (14)$$

Since, $\lambda_{qr}^{e} = 0$, the q-axis rotor current is given by

$$i_{qr}^{e} = -\frac{L_m}{L_{lr}} i_{qs}^{e} \qquad (15)$$

Combining **equations 13** and **15** gives

$$T_e = \frac{3}{2}\frac{P}{2}\frac{L_m}{L_{lr}}\lambda_{dr}^{e} i_{qs}^{e} \qquad (16)$$

The d-axis rotor flux is given by (refer Lecture 19)

$$\lambda_{dr}^{e} = L_{lr} i_{dr}^{e} + L_m \left(i_{ds}^{e} + i_{dr}^{e} \right) \qquad (17)$$

The **equation 7** gives $i_{dr}^{e} = 0$, hence **equation 17** can be written as

$$\lambda_{dr}^{e} = L_m i_{ds}^{e} \qquad (18)$$

Together, **equation 19** and **equation 21** suggest the generic rotor flux oriented control shown in **Figure 2**.

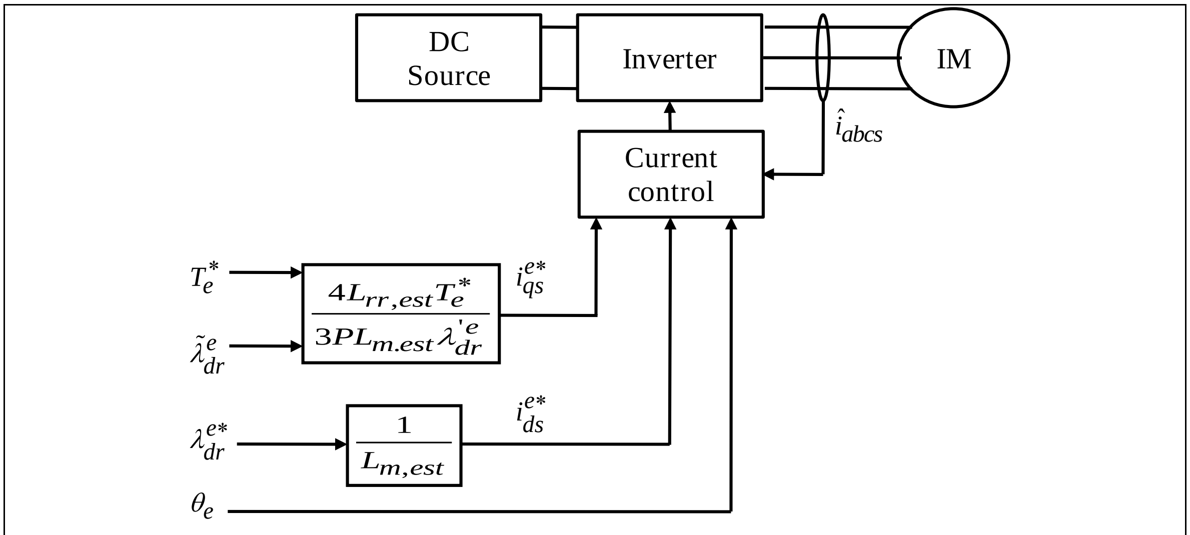

Figure 2: Generic Field Oriented Control (FOC) [1]

In **Figure 2** the variables of the form $x^*, \tilde{x}$ and $\hat{x}$ denote ***command***, ***measured*** and ***estimated*** values respectively. In case of ***parameters that are estimated***, a subscript "**est**" is used. The working of the controller is as follows:

i. Based on the torque command (T_e^*), the assumed values of the parameters and the estimated value of d-axis rotor flux $\hat{\lambda}_{dr}^s$ is used to formulate a q-axis stator current command i_{qs}^{s*}.
ii. The d-axis stator current command i_{ds}^{s*} is calculated such as to achieve a rotor flux command λ_{dr}^{s*} (using **equation 12**).s
iii. The q-axis and d-axis stator current command is then achieved using a current source control.

The above description of rotor flux oriented FOC is incomplete with determination of $\hat{\lambda}_{dr}^s$ and θ_s. The difference between ***direct*** and ***indirect*** FOC is in how these two variables are determined.

Direct Rotor Oriented FOC

In direct FOC, the position of the synchronous reference frame (θ_e) is determined based on the values of q-axis and d-axis rotor flux linkages in the stationary reference frame. The relation of flux linkages in synchronous reference frame and stationary reference frame is

$$\begin{bmatrix} \lambda_{qr}^e \\ \lambda_{dr}^e \end{bmatrix} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e \\ \sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} \lambda_{qr}^s \\ \lambda_{dr}^s \end{bmatrix} \tag{19}$$

where

λ_{qr}^s is the rotor q-axis flux linkage in stationary frame of reference

λ_{dr}^s is the rotor d-axis flux linkage in stationary frame of reference

In order to achieve $\lambda_{qr}^e = 0$, it is sufficient to define the position of the synchronous reference frame as

$$\theta_e = \tan^{-1}\left(\frac{\lambda_{qr}^s}{\lambda_{dr}^s}\right) + \frac{\pi}{2} \tag{20}$$

The difficulty with this approach is that λ_{qr}^s and λ_{dr}^s are not directly measurable quantities. However, they can be estimated using direct measurement of air gap flux. To measure the air gap flux, hall-effect sensors are placed in the air gap and used to measure the air-gap flux in q -axis and d -axis. Since the hall-effect sensors are stationary, the flux measured by them is in stationary reference frame. The flux measured by the sensors is the net flux in the air gap (combination of stator and rotor flux). The net flux in the air gap is given by:

$$\lambda_{qm}^s = L_m\left(i_{qs}^s + i_{qr}^s\right) \tag{21}$$

where

L_m is the magnetization inductance

From **equation 21**, the rotor q -axis current is obtained as

$$i_{qr}^s = \frac{\lambda_{qm}^s - L_m i_{qs}^s}{L_m} \tag{22}$$

The q -axis rotor flux linkage is given by

$$\lambda_{qr}^s = L_{lr} i_{qr}^s + L_m\left(i_{qs}^s + i_{qr}^s\right) \tag{23}$$

Substituting the rotor q -axis current from **equation 22** into **equation 23** gives

$$\lambda_{qr}^s = \frac{L_{lr}}{L_m}\lambda_{qm}^s - L_{lr} i_{qs}^s \tag{24}$$

An identical derivation for d -axis gives

$$\lambda_{dr}^{s} = \frac{L_{lr}}{L_{m}} \lambda_{dm}^{s} - L_{lr} i_{ds}^{s} \tag{25}$$

The implementation of this control strategy is shown in **Figure 3a** and **b**.

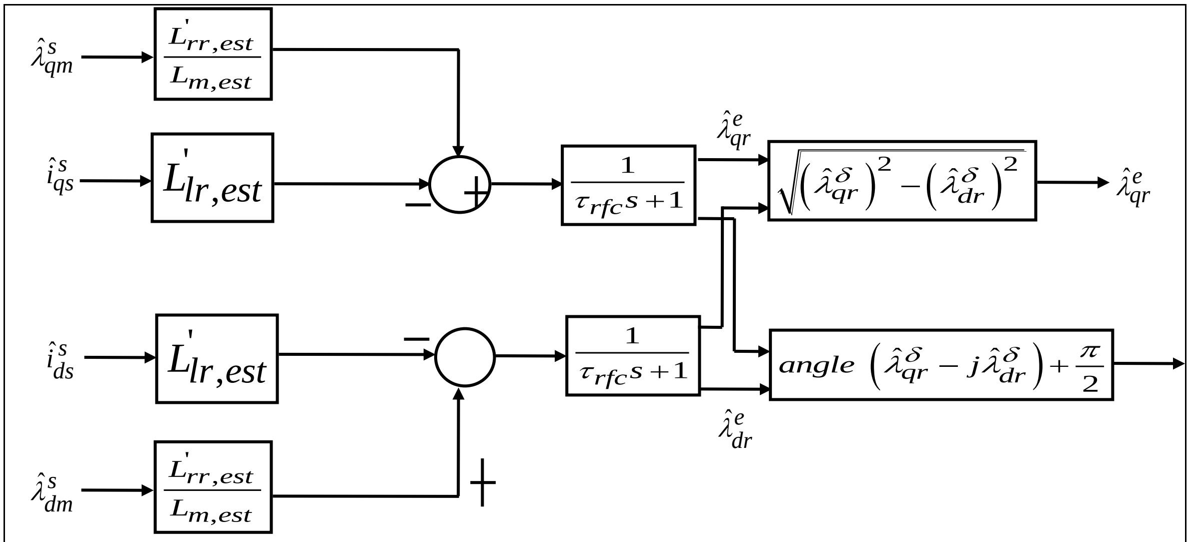

Figure 3a: Rotor flux calculation in Direct FOC [1]

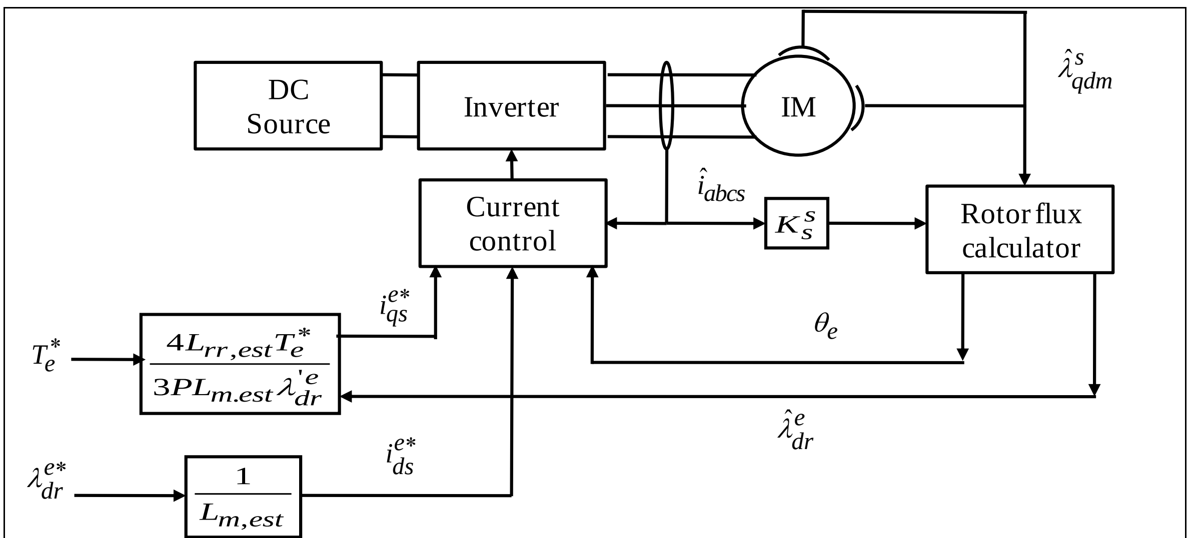

Figure 3b: Control strategy for Direct FOC [1]

Indirect Rotor Oriented FOC

The direct FOC is problematic and expensive due to use of hall-effect sensors. Hence, indirect FOC methods are gaining considerable interest. The indirect FOC methods are more sensitive to knowledge of the machine parameters but do not require direct sensing of the rotor flux linkages.

The q-axis rotor voltage equation in synchronous frame is

$$0 = r_r i_{qr}^e + \left(\omega_e - \omega_r\right)\lambda_{dr}^e + p\lambda_{qr}^e \tag{26}$$

Since $\lambda_{qr}^e = 0$ for direct field oriented control, **equation 26** becomes

$$\begin{aligned} &0 = r_r i_{qr}^e + \left(\omega_e - \omega_r\right)\lambda_{dr}^e \\ &\Rightarrow \omega_e = \omega_r - r_r \frac{i_{qr}^e}{\lambda_{dr}^e} \end{aligned} \tag{27}$$

Substituting the values of i_{qr}^e and λ_{dr}^e from **equation 15** and **18** respectively into **equation 27** gives

$$\omega_e = \omega_r + \frac{r_r}{L_{lr}}\frac{i_{qs}^e}{\lambda_{ds}^e} \tag{28}$$

From **equation 28** it can be observed that instead of establishing θ_e using the rotor flux as shown in **Figure 3**, it can be determined by integrating ω_e given by **equation 28** where ω_e is given as:

$$\omega_e = \omega_r + \frac{r_r}{L_{lr}}\frac{i_{qs}^{e*}}{\lambda_{ds}^{e*}} \tag{29}$$

The **equation 29** does satisfy the conditions of FOC. In order to check it, consider the rotor voltage equations for the q-axis and d-axis:

$$0 = r_r i_{qr}^e + \left(\omega_e - \omega_r\right)\lambda_{dr}^e + p\lambda_{qr}^e \tag{30}$$

$$0 = r_r i_{dr}^e + \left(\omega_e - \omega_r\right)\lambda_{qr}^e + p\lambda_{dr}^e \tag{31}$$

Substituting ω_e from **equation 29** into **equations 30** and **31** gives

$$0 = r_r i_{qr}^e + \frac{r_r}{L_{lr}}\frac{i_{qs}^{e*}}{i_{ds}^{e*}}\lambda_{dr}^e + p\lambda_{qr}^e \tag{32}$$

$$0 = r_r i_{dr}^e + \frac{r_r}{L_{lr}}\frac{i_{qs}^{e*}}{i_{ds}^{e*}}\lambda_{qr}^e + p\lambda_{dr}^e \tag{33}$$

Substituting the value of d -axis rotor flux from **equations 17** into **equation 33** gives

$$0 = r_r\left(\frac{\lambda_{qr}^e - L_m i_{qs}^{e*}}{L_{lr}}\right) i_{qr}^e + \frac{r_r}{L_{lr}}\frac{i_{qs}^{e*}}{i_{ds}^{e*}}\left(L_{lr} i_{dr}^{e*} + L_m i_{ds}^{e*}\right) + p\lambda_{qr}^e \tag{34}$$

$$0 = r_r i_{dr}^e - \frac{r_r}{L_{lr}}\frac{i_{qs}^{e*}}{i_{ds}^{e*}}\lambda_{qr}^e + p\left(L_{lr} i_{qr}^e + L_m i_{ds}^{e*}\right) \tag{35}$$

If the d -axis rotor current is held constant, then $p i_{dr}^{e*} = 0$ and rearranging **equations 34** and **35** gives

$$p\lambda_{qr}^e = -\frac{r_r}{L_{lr}}\lambda_{qr}^e - r_r\frac{i_{qs}^{e*}}{i_{ds}^{e*}} i_{dr}^e \tag{36}$$

$$p i_{dr}^e = -\frac{r_r}{L_{lr}}\lambda_{dr}^e + \frac{r_r}{Lllr}\frac{i_{qs}^{e*}}{i_{ds}^{e*}}\lambda_{dr}^e \tag{37}$$

The solution of **equations 37** and **38** will decay to zero (same argument as used for **equation 12**), hence λ_{qr}^e and i_{qr}^e will eventually become zero. In **Figure 4** the implementation of ***indirect FOC*** is shown and it is much simpler than the ***direct FOC***.

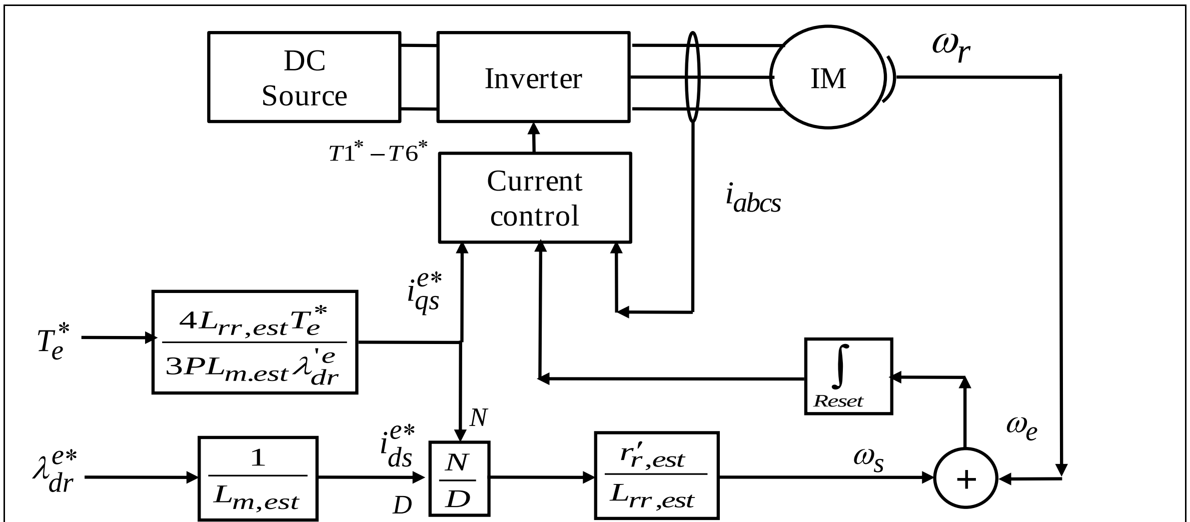

Figure 4: Indirect FOC [1]

References:

[1] P. C. Krause, O. Wasynczuk, S. D. Sudhoff, *Analysis of electric machinery*, IEEE Press, 1995

Suggested Reading

[1] R. Krishnan, *Electric motor drives: modeling, analysis, and control*, Prentice Hall, 2001